Lecture 09: Integral extensions

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<u>Def.</u> Suppose A is a subring of B, and $b \in B$. We say b is <u>integral</u> over A if $\exists\, a_0, \dots, a_n \in A$ s.t. $b^n + a_{n-1} b^{n-1} + \dots + a_0 = 0$.

(b is a zero of a <u>monic</u> polynomial in $A[x]$.)

- We say B/A is an <u>integral extension</u> if $A \subseteq B$ and $\forall\, b \in B$ b is integral over A.

<u>Ex.</u> D : UFD ; k : field of fractions of D;

$$\alpha \in k \text{ is integral over } D \iff \alpha \in D \qquad (*)$$

<u>Def.</u> We say an integral domain D is <u>integrally closed</u> if the above $(*)$ property holds.

<u>Proposition</u>. Suppose B/A is a ring extension. TFAE:

ⓐ $b \in B$ is integral over A.

ⓑ $A[b]$ is a finitely generated A-module.

ⓒ $\exists$ a subring C of B that contains $A[b]$ as a subring and C is a finitely generated A-module.

ⓓ $\exists$ a faithful $A[b]$-module M that is a finitely generated A-mod.

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Pf. (a) $\Rightarrow$ (b) Suppose b is a zero of the monic polynomial

$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in A[x]$.

Claim. $A[b] = A + Ab + \cdots + Ab^{n-1}$.

Pf of Claim. $\forall \alpha \in A[b]$, $\exists f(x) \in A[x]$, $\alpha = f(b)$. Since $p(x)$ is monic, we can divide $f(x)$ by $p(x)$; and so $\exists q(x), r(x)$ in $A[x]$ s.t. (1) $f(x) = p(x)\, q(x) + r(x)$, (2) $\deg r < \deg p$.

Hence $\alpha = f(b) = p(b)\, q(b) + r(b) = r(b) \in A + Ab + \cdots + Ab^{n-1}$.

(b) $\Rightarrow$ (c) it is clear; let $C = A[b]$.

(c) $\Rightarrow$ (d) it is clear; let $M = C$ (notice our rings are unital!)

(d) $\Rightarrow$ (a) Since M is an $A[b]$-mod, $b \in \mathrm{End}_A(M)$. Since M is a finitely generated A-module,

$\exists\, a_0, \ldots, a_{n-1} \in A$ s.t. $(b^n + a_{n-1}b^{n-1} + \cdots + a_0)\, M = 0$.

(We proved this result earlier and deduced Nakayama's lemma from this.) Since M is a faithful $A[b]$-module,

$$b^n + a_{n-1}b^{n-1} + \cdots + a_0 = 0. \qquad \blacksquare$$

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<u>Corollary</u>. Suppose B/A is a ring extension. Let

$$C := \{ b \in B \mid b \text{ is integral over } A \}.$$

Then C is a subring of B.

<u>Pf.</u> Suppose $b_1, b_2 \in C$. Then

$$A[b_1] = \sum_{j=0}^{n_1} A b_1^j \text{ and } A[b_2] = \sum_{j=0}^{n_2} A b_2^j.$$

Any element of $A[b_1, b_2]$ is of the form $\sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} a_{j_1 j_2} b_1^{j_1} b_2^{j_2}$

$$= \sum_{j_1=0}^{m_1} \underbrace{\left(\sum_{j_2=0}^{m_2} a_{j_1 j_2} b_2^{j_2} \right)}_{\text{in } A[b_2]} b_1^{j_1} = \sum_{j_1=0}^{m_1} \left(\sum_{i_2=0}^{n_2} a'_{j_1 i_2} b_2^{i_2} \right) b_1^{j_1}$$

$$= \sum_{i_2=0}^{n_2} \underbrace{\left(\sum_{j_1=0}^{m_1} a'_{j_1 i_2} b_1^{j_1} \right)}_{\text{in } A[b_1]} b_2^{i_2} = \sum_{i_2=0}^{n_2} \left(\sum_{i_1=0}^{n_1} a''_{i_1 i_2} b_1^{i_1} \right) b_2^{i_2}.$$

And so $A[b_1, b_2] = \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} A b_1^{j_1} b_2^{j_2}$ is a finitely generated A-module; and so by the previous proposition $A[b_1, b_2] \subseteq C$. Hence $b_1 b_2, b_1 - b_2 \in C$; therefore C is a subring. ∎

<u>Def.</u> • C is called the <u>algebraic closure of A in B</u>.

• We say A is <u>algebraically closed in B</u> if $C = A$.

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<u>Lemma</u>. Suppose B/A and C/B are integral extensions. Then C/A is an integral extension.

<u>Pf.</u> Let $c \in C$. Then $\exists\, b_0, \ldots, b_{n-1} \in B$ s.t. $c^n + b_{n-1} c^{n-1} + \cdots + b_0 = 0$.

Since b_i are integral over A, for $m \gg 1$ $A[b_i] = \sum_{j=0}^{m} A\, b_i^j$.

Hence $A[b_0, \ldots, b_{n-1}] = \sum_{0 \le j_0, \ldots, j_{n-1} \le m} A\, b_0^{j_0} \cdots b_{n-1}^{j_{n-1}}$. And so

$$A[b_0, \ldots, b_{n-1}, c] = \sum_{i=0}^{n-1} \sum_{0 \le j_0, \ldots, j_{n-1} \le m} A\, b_0^{j_0} \cdots b_{n-1}^{j_{n-1}}\, c^i$$

is a finitely generated A-module. And so c is integral over A. ∎

<u>Corollary</u>. Let B/A be a ring extension. Then the integral closure of A in B is integrally closed in B.

<u>Pf.</u> Let C be the integral closure of A in B, and C' be the integral closure of C in B. Then C/A and C'/C are integral extensions. And so by Lemma, C'/A is an integral extension, which implies $C' \subseteq C$; and claim follows. ∎

<u>Corollary</u>. Suppose $k/\mathbb{Q}$ is a finite extension. Let $\mathcal{O}_k$ be the integral closure of $\mathbb{Z}$ in k. Then $\mathcal{O}_k$ is integrally closed.

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<u>Pf.</u> By the previous Corollary, $\mathcal{O}_k$ is integrally closed in k, and the field of fractions of $\mathcal{O}_k$ is a subfield of k (in fact we will see that it is k). And so $\mathcal{O}_k$ is integrally closed in its field of fractions; and claim follows. ■

In the next lecture we will show:

- $f: A \hookrightarrow B$ integral $\Rightarrow$
  - ⓐ $f^*: \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is onto.
  - ⓑ $\dim (f^*)^{-1}(\mathfrak{p}) = 0 \quad \forall \mathfrak{p} \in \mathrm{Spec}(A)$.
  - ⓒ $\dim A = \dim B$ (we deduce this using Going-Up theorem.)