Lecture 13: Going-Down theorem: 2nd proof

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Theorem. . A: integral domain, integrally closed, with field of fract. F.

E/F: normal field extension.

B: integral closure of A in E. $(f: A \hookrightarrow B)$

$\Rightarrow \forall \mathfrak{p} \in \operatorname{Spec}(A)$, $\operatorname{Aut}(E/F) \curvearrowright (f^*)^{-1}(\mathfrak{p})$ transitively.

- We proved that $\operatorname{Aut}(E/F) \curvearrowright B$ and $(f^*)^{-1}(\mathfrak{p})$.
- Let $F' := E^{\operatorname{Aut}(E/F)}$. Then E/F' is Galois and F'/F is purely inseparable.

To see the second part, notice that since E/F is normal, $\operatorname{Aut}(E/F)$ acts transitively on zeros of $\min(\alpha; F)$. And so for $\alpha \in \operatorname{Fix}(G)$, $g(x) := \min(\alpha; F) = (x-\alpha)^m$ for some m. If $\operatorname{char} F = 0$, then $\operatorname{Fix}(G) = F$. If $\operatorname{char}(F) = p > 0$ and $m = p^\ell k$, then $g(x) = (x^{p^\ell} - \alpha^{p^\ell})^k \in F[x]$. And so $h(x) = x^{p^\ell \cdot k} - \alpha^{p^\ell \cdot k} \in F[x]$ and α is a zero of $h(x)$. Hence $h(x) = g(x)$; thus $(y - \alpha^{p^\ell})^k = y^k - \alpha^{p^\ell \cdot k}$ which implies $k=1$ as otherwise the LHS has multiple zeros, but the RHS does not.

Let A' be the integral closure of A in $\operatorname{Fix}(G)$. Then $\exists \mathfrak{p}' \in \operatorname{Spec} A'$ s.t. $\mathfrak{p}' \cap A = \mathfrak{p}$. $\forall \alpha \in \mathfrak{p}'$, $\alpha^{p^m} \in A$ for some $m \in \mathbb{Z}^{\geq 0}$. Hence

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$\forall \alpha\in\mathfrak{P}'$, $\alpha^{p^m}\in\mathfrak{p}$ for some $m\in\mathbb{Z}^{\geq 0}$. Let

$$\tilde{\mathfrak{P}} := \{\alpha\in A' \mid \exists m\in\mathbb{Z}^{\geq 0},\ \alpha^{p^m}\in\mathfrak{p}\}.$$

<u>Claim.</u> $\tilde{\mathfrak{P}}\in \operatorname{Spec} A'$ and $\tilde{\mathfrak{P}}\cap A=\mathfrak{p}$.

<u>Pf of Claim.</u> • $\alpha_1,\alpha_2\in\tilde{\mathfrak{P}} \Rightarrow \exists\, m_i,\ \alpha_i^{p^{m_i}}\in\mathfrak{p}$. Let m be $\max\{m_1,m_2\}$. Then $(\alpha_1+\alpha_2)^{p^m}=\alpha_1^{p^m}+\alpha_2^{p^m}\in\mathfrak{p}$.

• $\alpha\in A'$, $\beta\in\tilde{\mathfrak{P}} \Rightarrow \exists\, m\in\mathbb{Z}^{\geq 0}$, $\alpha^{p^m}\in F$ and $\beta^{p^m}\in\mathfrak{p}$.

Since α is integral over A and A is integrally closed, $\alpha^{p^m}\in A$. Hence $(\alpha\beta)^{p^m}=\alpha^{p^m}\cdot\beta^{p^m}\in\mathfrak{p}$. And so $\alpha\beta\in\tilde{\mathfrak{P}}$.

• $\alpha_1\alpha_2\in\tilde{\mathfrak{P}}$ for $\alpha_i\in A' \Rightarrow \exists\, m_i$ s.t. $\alpha_i^{p^{m_i}}\in F$.

Since α_i is integral over A and A is integrally closed, $\alpha_i^{p^{m_i}}\in A \Rightarrow \alpha_i^{p^m}\in A$ for $m\geq\max\{m_1,m_2\}$. And so $\exists m$ $(\alpha_1\alpha_2)^{p^m}\in\mathfrak{p}$ and $(\alpha_i)^{p^m}\in A$. As $\mathfrak{p}$ is prime, either $\alpha_1^{p^m}\in\mathfrak{p}$ or $\alpha_2^{p^m}\in\mathfrak{p}$. Hence either $\alpha_1\in\tilde{\mathfrak{P}}$ or $\alpha_2\in\tilde{\mathfrak{P}}$.

• $\alpha\in\tilde{\mathfrak{P}}\cap A \Rightarrow \alpha^{p^m}\in\mathfrak{p} \Rightarrow \alpha\in\mathfrak{p}$. ∎

Hence there is only $\tilde{\mathfrak{P}}\in\operatorname{Spec} A'$ that is over $\mathfrak{p}$.

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Hence, as $\mathrm{Aut}(E/F)=\mathrm{Aut}(E/F')$, after changing A with A' and $\mathfrak{p}$ with $\tilde{\mathfrak{p}}$, we can and will assume E/F is a Galois extension.

Notice that if $\mathrm{char}(F)=0$, then $F=F'$ and the above argu. is not needed

<u>Case 1</u>. $[E:F]<\infty$ and Galois.

<u>Pf.</u> Suppose to the contrary that $\exists\, \mathfrak{q}_2 \in (f^*)^{-1}(\mathfrak{p}) \setminus \mathrm{Gal}(E/F)\cdot \mathfrak{q}_1$.

Since dim. of $(f^*)^{-1}(\mathfrak{p})$ is zero, $\mathfrak{q}_2 \not\subseteq \sigma(\mathfrak{q}_1)$ $\forall \sigma\in \mathrm{Aut}(E/F)$.

As $|\mathrm{Gal}(E/F)|<\infty$, $\mathfrak{q}_2 \not\subseteq \bigcup_{\sigma\in \mathrm{Gal}(E/F)} \sigma(\mathfrak{q}_1)$. Suppose $\alpha\in \mathfrak{q}_2 \setminus \bigcup_\sigma \sigma(\mathfrak{q}_1)$

Then $\prod_{\sigma\in\mathrm{Gal}(E/F)} \sigma(\alpha)\in F$. At the same time $\sigma(\alpha)\in B$ $(\forall\sigma)$

and $\alpha\in\mathfrak{q}_2$; and so $N_{E/F}(\alpha)$ is integral over A and in $\mathfrak{q}_2$. As

A is integrally closed, we deduce $N_{E/F}(\alpha)\in A\cap\mathfrak{q}_2=\mathfrak{p}$.

$\Rightarrow \prod_{\sigma\in\mathrm{Gal}(E/F)} \sigma(\alpha)\in\mathfrak{p}\subseteq\mathfrak{q}_1 \Rightarrow \sigma(\alpha)\in\mathfrak{q}_1$ for some σ

$\Rightarrow \alpha\in\sigma^{-1}(\mathfrak{q}_1)$ for some σ, which is a contradiction.

<u>Case 2</u>. The general case.

$\exists\, E_i\subseteq E$, $i\in I$ s.t. (1) E_i/F finite Galois

(2) $\forall i,j$, $\exists k$ s.t. $E_k\supseteq E_i\cup E_j$ (3) $E=\bigcup_{i\in I} E_i$.

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- For $\mathfrak{q}_1, \mathfrak{q}_2 \in \operatorname{Spec}(B)$ s.t. $\mathfrak{q}_1 \cap A = \mathfrak{q}_2 \cap A = \mathfrak{p}$, and any $i \in I$, by the finite Galois case,

$\Gamma_i := \{\sigma \in \operatorname{Gal}(E_i/F) \mid \sigma(\mathfrak{q}_1 \cap E_i) = \mathfrak{q}_2 \cap E_i\} \neq \emptyset$; and

clearly, for $E_i \subseteq E_j$ and $\sigma \in \Gamma_j$, $\sigma|_{E_i} \in \Gamma_i$. Hence $\varprojlim \Gamma_i \neq \emptyset$ and gives us an element σ of $\operatorname{Gal}(E/F)$ s.t. $\forall i$ $\sigma(\mathfrak{q}_1 \cap E_i) = \mathfrak{q}_2 \cap E_i$. As $\mathfrak{q}_j = \bigcup_{i \in I} (\mathfrak{q}_j \cap E_i)$, we get $\sigma(\mathfrak{q}_1) = \mathfrak{q}_2$. ■

2nd proof of the Going-Down Theorem.

Let F be the field of fractions of A and E be the field of fractions of B. Let K be the normal closure of E over F, and C be the integral closure of A in K. So C/B is an integral extension and $\operatorname{Spec} C \to \operatorname{Spec} B \to \operatorname{Spec} A$ are onto. Suppose $\mathfrak{P}_0, \mathfrak{P}_1 \in \operatorname{Spec} C$ s.t. $\mathfrak{P}_0 \cap A = \mathfrak{p}_0$ and $\mathfrak{P}_1 \cap B = \mathfrak{q}_1$. By the Going-Up theorem $\exists\, \mathfrak{P}_1' \in \operatorname{Spec} C$ s.t. $\mathfrak{P}_1' \cap B = \mathfrak{q}_1$. As $\mathfrak{P}_1$ and $\mathfrak{P}_1'$ are two primes in the fiber of $\mathfrak{q}_1$, $\exists\, \sigma \in \operatorname{Aut}(K/E)$ s.t. $\mathfrak{P}_1 = \sigma(\mathfrak{P}_1')$. Hence $\mathfrak{P}_0 \subseteq \mathfrak{P}_1'$ implies $\sigma(\mathfrak{P}_0) \subseteq \mathfrak{P}_1$, and

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$\sigma(\mathfrak{P}_0) \cap A = \mathfrak{P}_0 \cap A = \mathfrak{p}_0$. Hence $\mathfrak{q}_0 := B \cap \sigma(\mathfrak{P}_0) \subseteq \mathfrak{q}_1$, $\mathfrak{q}_0 \in \operatorname{Spec} B$

and $\mathfrak{q}_0 \cap A = \mathfrak{p}_0$. $\blacksquare$