Lecture 14: Integral+integrally closed => open

Friday, April 20, 2018 12:13 AM

Theorem. A: integral domain, integrally closed

B: integral domain, $f: A \hookrightarrow B$ integral injection

$\Rightarrow f^*: \operatorname{Spec} B \to \operatorname{Spec} A$ is open.

In fact, for $b \in B$, suppose $\min(b; F) := x^n + a_{n-1}x^{n-1} + \cdots + a_0$.

Then $f^*(D(b)) = \bigcup_{i=0}^{n-1} D(a_i) = \operatorname{Spec} A \setminus V(a_0, a_1, \ldots, a_{n-1})$.

Pf. • Suppose $\underbrace{f^*(\mathfrak{q})}_{\mathfrak{p}} \notin \bigcup_{i=0}^{n-1} D(a_i)$ for some $\mathfrak{q} \in D(b)$. So $a_i \in \mathfrak{p}$,

which implies b is integral over $\mathfrak{p}$. Hence $b \in \sqrt{\mathfrak{p}^e} \subseteq \mathfrak{q}$,

which implies $\mathfrak{q} \notin D(b)$; this is a contradiction.

• For $\mathfrak{p} \in \bigcup_{i=0}^{n-1} D(a_i)$, suppose $(f^*)^{-1}(\mathfrak{p}) \cap D(b) = \emptyset$. ⊛

Let F be the field of fractions of A, E be the field of fractions of B, and K be the normal closure of E over F. Let C be the integral closure of A in K, and $g: A \hookrightarrow C$. Then ⊛ implies $(g^*)^{-1}(\mathfrak{p}) \cap D(b) = \emptyset$. Let $\mathfrak{P} \in (g^*)^{-1}(\mathfrak{p})$. Hence $\forall \sigma \in \operatorname{Aut}(K/F)$, $b \in \sigma(\mathfrak{P})$; which implies $\forall \sigma \in \operatorname{Aut}(K/F)$, $\sigma(b) \in \mathfrak{P}$. Hence all the zeros of $\min(b; E)$ are in $\mathfrak{P}$. This implies $a_i \in A \cap \mathfrak{P} = \mathfrak{p}$ which is a contradiction. ∎

Lecture 14: Integral closure; Noetherian; separablity

Friday, April 27, 2018 8:40 AM

Next we would like to extend the Noetherian property to certain integral extensions:

<u>Proposition</u>. A: integral domain, integrally closed;

F: field of fractions of A;

E/F: separable finite field extension;

B: the integral closure of A in E;

$\Rightarrow \exists\, e_1, \ldots, e_n \in E$ s.t. $B \subseteq Ae_1 + \cdots + Ae_n$.

In particular, if A is Noetherian, then B is Noetherian.

We start with a lemma from field theory:

<u>Lemma</u>. Let E/F be a finite separable field extension. For any $e \in E$, let $l_e \in \mathrm{End}_F(E)$, $l_e(e') = ee'$. Then, over $\overline{F}$, l_e is similar to $\mathrm{diag}(\sigma_1(e), \ldots, \sigma_n(e))$ where $\{\sigma_1, \ldots, \sigma_n\} = \mathrm{Hom}_F(E, \overline{F})$.

In particular, $\mathrm{tr}(l_e) = \sum \sigma_i(e)$ and $\det(l_e) = \prod \sigma_i(e)$.

<u>Pf</u>. Since E/F is a finite separable extension, $\exists\, \alpha \in E$ s.t.

$E = F[\alpha] \simeq F[x]/\langle \min(\alpha; F)\rangle$; and $\min(\alpha; F) = (x-\alpha_1)\cdots(x-\alpha_j)$

where $\alpha_i \neq \alpha_j \in \overline{F}$ (and $\alpha_1 = \alpha$). Hence $E \otimes_F \overline{F} \simeq \overline{F}[x]/\langle \prod (x-\alpha_i)\rangle$.

Lecture 14: Separablity and trace

Monday, April 30, 2018 8:46 AM

And by the Chinese Remainder Theorem, $E \otimes_F \overline{F} \simeq \bigoplus_{i=1}^{n} \overline{F}[x]/\langle x-\alpha_i \rangle$

$\Rightarrow E \otimes_F \overline{F} \simeq \bigoplus_{i=1}^{n} \overline{F}$; and so $e \otimes 1 \mapsto (\sigma_1(e), \dots, \sigma_n(e))$.

$\alpha \otimes 1 \mapsto (\alpha_1, \dots, \alpha_n)$

Therefore $\ell_e \otimes \mathrm{id}$ in the standard basis of $\bigoplus_{i=1}^{n} \overline{F}$ is

$\mathrm{diag}(\sigma_1(e), \dots, \sigma_n(e))$. And claim follows. ■

<u>Proposition</u>. Suppose E/F is a finite separable field extension.

$\forall e \in E$, let $T_{E/F}(e) := \mathrm{tr}(\ell_e)$. Let $f: E \times E \to F$,

$f(e_1, e_2) := T_{E/F}(e_1 e_2)$. Then f is a non-degenerate bilinear form.

<u>Pf.</u> Since $T_{E/F}: E \to F$ is F-linear, f is a bilinear form.

To show it is non-degenerate one has to take an F-basis $\{e_1, \dots, e_n\}$ of E and show $\det[f(e_i, e_j)] \neq 0$.

$f(e_i, e_j) = T_{E/F}(e_i e_j) = \sum_{k=1}^{n} \sigma_k(e_i e_j)$ by the previous lemma where $\{\sigma_1, \dots, \sigma_n\} = \mathrm{Hom}_F(E, \overline{F})$. Hence

$[f(e_i, e_j)] = [\sum_{k=1}^{n} \sigma_k(e_i)\, \sigma_k(e_j)] = [\sigma_j(e_i)]\, [\sigma_j(e_i)]^t$. And so

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Wednesday, May 2, 2018 12:08 AM

$$\det[f(e_i, e_j)] = \det([\sigma_j(e_i)][\sigma_j(e_i)]^t) = \det([\sigma_j(e_i)])^2 .$$

As in the proof of the previous lemma

$$E \otimes_F \overline{F} \longrightarrow \bigoplus_{i=1}^{n} \overline{F}, \quad e \otimes 1 \longmapsto (\sigma_1(e), \ldots, \sigma_n(e))$$

is an $\overline{F}$-isomorphism. Hence

$$(\sigma_1(e_1), \ldots, \sigma_n(e_1)), \; (\sigma_1(e_2), \ldots, \sigma_n(e_2)), \ldots, (\sigma_1(e_n), \ldots, \sigma_n(e_n))$$

are $\overline{F}$-linearly independent as

$$E = \bigoplus_{i=1}^{n} F e_i \quad \text{implies} \quad E \otimes_F \overline{F} = \bigoplus_{i=1}^{n} (e_i \otimes \overline{F}) .$$

And so $\det[\sigma_j(e_i)] \neq 0$. ∎

<u>Lemma</u>. Suppose F is the field of fractions of A and E/F is an algebraic extension. Let B be the integral closure of A in E. Then $E = (A \setminus \{0\})^{-1} B$.

(We will prove this in the next lecture.)