Lecture 17: Integral closure and valuation rings

Monday, May 7, 2018 10:07 AM

In the previous lecture we proved the following technical theorem:

Ω: algebraically closed field;

A_0 : subring of a field F; $\phi_0 : A_0 \to \Omega$ ring hom.;

$$\Sigma := \left\{ (A,\phi) \;\middle|\; \underset{\text{subring}}{A_0 \subseteq A \subseteq F}\ ;\ \begin{tikzcd} A \arrow[dr, "\phi"] & \\ A_0 \arrow[u, hook] \arrow[r, "\phi_0"'] \arrow[ur, phantom, "\circlearrowleft"] & \Omega \end{tikzcd} \right\};$$

$(A_1, \phi_1) \preccurlyeq (A_2, \phi_2)$ if $A_1 \subseteq A_2$ and $\phi_2|_{A_1} = \phi_1$.

Then Σ has a maximal element; if (B,θ) is maximal in Σ, then B is a valuation ring, its field of fractions is F, and $\ker\theta$ is the maximal ideal of B.

<u>Proposition</u>. A: integral domain with field of fractions F

$$\text{integral closure of } A \text{ in } F = \bigcap_{\substack{A \subseteq B \subseteq F \\ B \text{ valuation ring}}} B$$

<u>Pf.</u>. Since any valuation ring is integrally closed and by the tech. theorem $\exists A \subseteq B \subseteq F$, B valu. ring, RHS $\subseteq$ LHS.

- Suppose $f \in F$ is not integral over A. So $f \notin A[f^{-1}]$; this implies $\exists\, \mathfrak{m} \in \operatorname{Max}(A[f^{-1}])$ s.t. $f^{-1} \in \mathfrak{m}$. Let Ω be an alg.

Lecture 17: Lifting ring homomorphisms

Monday, May 7, 2018 8:35 AM

closure of $A[f^{-1}]/\mathfrak{m}$, and $\underbrace{A[f^{-1}] \to A[f^{-1}]/\mathfrak{m} \hookrightarrow \Omega}_{\phi_0}$.

Then by the technical theorem $\exists$ a valuation ring B and

$\theta : B \to \Omega$ s.t. $A[f^{-1}] \subseteq B$ and $\ker \theta \supseteq \mathfrak{m} \ni f^{-1}$.

Hence $f \notin B$. $\blacksquare$

The 2^{nd} important consequence of the technical theorem is:

<u>Theorem.</u> A : integral domain

B : integral domain ; f.g. A-alg. ; $A \subseteq B$.

$b_0 \in B \setminus \{0\}$.

$\Rightarrow \exists\ a_0 := a_0(b_0) \in A$ s.t. $\forall\ \phi \in \mathrm{Hom}(A, \Omega)$, $\phi(a_0) \neq 0$

$\exists\ \theta \in \mathrm{Hom}(B, \Omega)$ s.t. $\theta|_A = \phi$ and $\theta(b_0) \neq 0$.

<u>Pf.</u> We proceed by induction on the number of generators of B as an A-algebra. So it is enough to prove the case $B = A[\beta]$.

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, hook] & \Omega \\ B \arrow[ur, dashed, "\theta"'] & \end{tikzcd}$$

<u>Case 1.</u> β is transcendental over A.

Then $b_0 = c_n \beta^n + c_{n-1}\beta^{n-1} + \cdots + c_0$ where $c_i \in A$. Let $a_0 := c_n$.

Lecture 17: Extending ring homomorphism

Monday, May 7, 2018 8:47 AM

Then $\phi(c_n)t^n+\phi(c_{n-1})t^{n-1}+\cdots+\phi(c_0)$ is a non-zero poly. in $\Omega[t]$ if $\phi(a_0)\neq 0$. As $|\Omega|=\infty$, $\exists\ \omega_0\in\Omega$ which is not a zero of this poly. Then

$\theta(\sum d_i\beta^i) := \sum \phi(d_i)\omega_0^i$ satisfies the needed conditions.

<u>Case 2.</u> β is algebraic over A.

. Then $B=A[\beta]/A$ is algebraic. So $\exists\ a_i'$ and $a_i''\in A$ s.t.

(I) $a_n'\beta^n+a_{n-1}'\beta^{n-1}+\cdots+a_0'=0$ and

(II) $a_m'' b_0^{-m}+a_{m-1}' b_0^{-(m-1)}+\cdots+a_0''=0$.

Let $a_0:=a_n'a_m''$. If $\phi(a_n'a_m'')\neq 0$ for some

$\phi:A\to\Omega$, then ϕ has a lift $\hat{\phi}: A[1/a_n'a_m'']\to\Omega$.

Consider $A[1/a_n'a_m'']$ as a subring of the field E of fractions of B, and use the technical theorem to deduce:

$\exists$ a valuation ring C with field of fractions E, and a lift

$\tilde{\phi}: C\to\Omega$ of $\hat{\phi}$.

By (I) and (II), β and b_0^{-1} are integral over $A[1/a_n'a_m'']$. And so

Lecture 17: Extending ring homomorphisms

Tuesday, May 8, 2018 11:37 PM

β and b_0^{-1} are integral over C. Since C is integrally closed, $\beta, b_0^{-1} \in C$.

$\Rightarrow$ ① $B \subseteq C$ ② $b_0 \in C^\times \Rightarrow \tilde{\Phi}|_B : B \to \Omega$ is a lift of ϕ

and $\tilde{\Phi}(b_0) \neq 0$. ∎

<u>Theorem</u> (1st version of Hilbert's Nullstellensatz)

k: field. B: f.g. k-algebra.

If B is a field, then $B/_k$ is a finite extension.

<u>Pf.</u> Let Ω be an algebraic closure of k, and $\phi : k \hookrightarrow \Omega$ be an embedding. Let $b_0 := 1$. Then $\exists\, a_0 \in k$, $a_0 \neq 0$ s.t. if $\phi(a_0) \neq 0$, then ϕ has a lift $\hat{\Phi} : B \to \Omega$. But, since ϕ is an embedding, $\phi(a_0) \neq 0$. So $\exists$ a lift $\hat{\Phi} : B \to \Omega$ of $\phi : k \hookrightarrow \Omega$.

Since B is a field, $\hat{\Phi}$ is an embedding. And so $B/_k$ is an algebraic extension. Since B is a f.g. k-algebra, $B/_k$ is a finite extension. ∎