Lecture 19: 4th version of Hilbert's Nullstellensatz

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In the previous lecture we were proving:

<u>Theorem</u> (Our 4^{th} version of Hilbert's Nullstellensatz)

Suppose $\mathfrak{a} \underset{\neq}{\trianglelefteq} k[x_1,\ldots,x_n]$. Then $I(X(\mathfrak{a}))=\sqrt{\mathfrak{a}}$.

<u>Pf.</u> We have already proved that $\sqrt{\mathfrak{a}} \subseteq I(X(\mathfrak{a}))$. Suppose to the contrary that $\exists f \in I(X(\mathfrak{a}))\setminus\sqrt{\mathfrak{a}}$. Then $S_f \cap \mathfrak{a} = \emptyset$ where $S_f=\{1,f,f^2,\ldots\}$.

<u>Lemma</u>. Suppose D is an integral domain, and $d_0\in D\setminus\{0\}$. Then

$$S_{d_0}^{-1}D \simeq D[x]/\langle d_0x-1\rangle .$$

<u>Pf</u>. Let $\tilde{\Phi}: D[x] \to S_{d_0}^{-1}D$, $\tilde{\Phi}(p(x)) := p(1/d_0)$.

Then $d_0x-1 \in \ker\tilde{\Phi}$. So $\exists\ \phi: D[x]/\langle d_0x-1\rangle \to S_{d_0}^{-1}D$,

$\phi(p(x)+\langle d_0x-1\rangle) := p(d_0)$; and clearly ϕ is onto.

- Since $\bar{x}.\bar{d_0}=\bar{1}$ in $D[x]/\langle d_0x-1\rangle$, $\overline{S_{d_0}}$ consists of units in $D[x]/\langle d_0x-1\rangle$. Hence by the universal property of localization, $\exists\ \theta: S_{d_0}^{-1}D \to D[x]/\langle d_0x-1\rangle$, $\theta(\frac{d}{d_0^n}) = dx^n+\langle d_0x-1\rangle$, a ring hom.

Clearly θ and ϕ are inverse of each other. ∎

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By the above lemma, $S_f^{-1} k[x_1,\dots,x_n] \simeq k[x_1,\dots,x_n,x_{n+1}]/\langle x_{n+1}f-1\rangle$.

Hence $0 \neq S_f^{-1}k[x_1,\dots,x_n]/S_f^{-1}\mathfrak{a} \simeq k[x_1,\dots,x_n,x_{n+1}]/\langle x_{n+1}f-1\rangle + \mathfrak{a}[x_{n+1}]$

Therefore

$$\mathfrak{a}[x_{n+1}] + \langle x_{n+1}\cdot f - 1\rangle \text{ is a proper ideal.}$$

Hence by the 3rd version of Hilbert's Nullstellensatz,

$\exists\ (\underbrace{p_1,\dots,p_n}_{\vec{p}}, p_{n+1}) \in X(\mathfrak{a}[x_{n+1}] + \langle x_{n+1}\cdot f-1\rangle)$. And so

$\vec{p} \in X(\mathfrak{a})$ and $p_{n+1}\cdot f(\vec{p}) - 1 = 0$.

$$\left.\begin{array}{l}\vec{p}\in X(\mathfrak{a})\\ f\in I(X(\mathfrak{a}))\end{array}\right\} \Rightarrow \left.\begin{array}{l} f(\vec{p})=0 \\ p_{n+1}\cdot f(\vec{p})-1=0\end{array}\right\} \Rightarrow 1=0$$ which is a contradiction. ■

<u>Def.</u> A ring A is called a <u>Jacobson ring</u> if

$$\forall \mathfrak{p}\in \operatorname{Spec} A,\quad \mathfrak{p} = \bigcap_{\substack{\mathfrak{p}\subseteq \mathfrak{m} \\ \mathfrak{m}\in \operatorname{Max} A}} \mathfrak{m}.$$

<u>Theorem.</u> Any f.g. k-algebra is a Jacobson ring if k is a algebraically closed.

Pf. Since $A = k[x_1,\dots,x_n]/\mathfrak{a}$, Spec A can be identified with $V(\mathfrak{a})$,

and Max A can be identified with Max $k[x_1,\dots,x_n] \cap V(\mathfrak{a})$,

it is enough to prove $k[x_1,\dots,x_n]$ is a Jacobson ring.

$$\forall\, \mathfrak{p} \in k[x_1,\dots,x_n],\ \mathfrak{p} = \sqrt{\mathfrak{p}} = I(X(\mathfrak{p})) = \bigcap_{p \in X(\mathfrak{p})} \mathfrak{m}_p$$

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$$= \bigcap_{\substack{\mathfrak{p} \subseteq \mathfrak{m} \\ \mathfrak{m} \in \operatorname{Max} A}} \mathfrak{m} \quad . \blacksquare$$

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Corollary. Suppose k is algebraically closed and A is a f.g. k-alg.

Then $J(A)^n = 0$ for some $n \in \mathbb{Z}^+$.

Pf. By the above theorem A is a Jacobson ring. So $J(A) = \mathrm{Nil}(A)$.

$$\Big(\mathrm{Nil}(A) = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \ \bigcap_{\substack{\mathfrak{m} \in \operatorname{Max} A \\ \mathfrak{m} \mid \mathfrak{p}}} \mathfrak{m} = \bigcap_{\mathfrak{m} \in \operatorname{Max} A} \mathfrak{m}$$

$$= J(A). \Big)$$

Since A is a f.g. k-algebra, by Hilbert's basis theorem A is Noetherian. Hence $\mathrm{Nil}(A)$ is a f.g. ideal. And so $\mathrm{Nil}(A)^n = 0$

for some $n\in\mathbb{Z}^+$; and claim follows. ∎

Remark. The above argument implies:

if A is a Noetherian Jacobson ring, then $J(A)^n=0$ for some $n\in\mathbb{Z}^+$.

Remark. In the previous theorem, the algebraically closed assumption is not necessary.

Another important result in the theory of f.g. k-algebras is Noether's normalization lemma:

Theorem. Suppose k is a field and A is a finitely generated k-algebra. Then $\exists\, x_1,\dots,x_n\in A$ s.t.

(1) $x_1,\dots,x_n$ are algebraically independent over k.

(2) A is integral over $k[x_1,\dots,x_n]$.

Lemma. For any positive integer M, let

$$\Phi_M^{\pm}: k[x_1,\dots,x_n]\longrightarrow k[x_1,\dots,x_n],\quad \Phi_M^{\pm}(x_n)=x_n$$
$$\Phi_M^{\pm}(x_i)=x_i\pm x_n^{M^i}.$$

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Then $\phi_M^+ \circ \phi_M^- = \phi_M^- \circ \phi_M^+ = \mathrm{id}_{k[x_1,\dots,x_n]}$; and so $\phi_M^\pm$ are automorphisms of $k[x_1,\dots,x_n]$.

Clear. $\square$

Corollary. For any $f \in k[x_1,\dots,x_n]$, $\exists\ \phi \in \mathrm{Aut}(k[x_1,\dots,x_n])$ s.t. the leading coeff. of $\phi(f)$ viewed as an element of $(k[x_1,\dots,x_{n-1}])[x_n]$ is in $k^\times$.

Pf. Suppose $M > \deg f$; then

$$\phi_M(f) = \sum a_{i_1\cdots i_n} (x_1 + x_n^M)^{i_1} \cdots (x_{n-1} + x_n^{M^{n-1}})^{i_{n-1}} \cdot x_n^{i_n}$$

$$= \sum a_{i_1\cdots i_n} x_n^{i_n + i_1\cdot M + i_2\cdot M^2 + \cdots + i_{n-1}\cdot M^{n-1}} + \text{terms of lower degree}$$

Since $\underline{0 \le i_j < M}$, $i_n + i_1 M + \cdots + i_{n-1} M^{n-1}$ uniquely determines $i_1,\dots,i_n$.

And so all these terms are distinct and claim follows. $\blacksquare$

Remark. When k is infinite, one can choose ϕ among functions of the form $\begin{cases} \phi(x_i) = x_i + \lambda x_n & \text{if } 1 \le i \le n-1 \\ \phi(x_n) = x_n \end{cases}$.

(why?)

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Pf of Noether normalization.

We proceed by induction on the number of generators of A.

Base case. $A = k[\alpha]$.

If α is algebraic over k, then A is integral over k ✓

If α is not algebraic over k ✓

(We will continue in the next lecture.)