Lecture 22: Artinian rings

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Lemma. A: Artinian $\Rightarrow$ $\dim A = 0$.

Pf. We know $\dim A = 0 \Leftrightarrow \operatorname{Spec} A = \operatorname{Max} A$. Suppose $\mathfrak{p} \in \operatorname{Spec} A$, we have to show $\bar{A} := A/\mathfrak{p}$ is a field. Suppose $x \in \bar{A} \setminus \{0\}$, and consider $\langle x \rangle \supseteq \langle x^2 \rangle \supseteq \cdots$. Since A is Artinian, so is $\bar{A}$. Hence $\exists\, n \in \mathbb{Z}^+$, $x^n \in \langle x^{n+1} \rangle$; and so $\exists\, y \in \bar{A}$ s.t. $x^n = x^{n+1} y$. Since $\bar{A}$ is an integral domain, it has the cancellation property. Hence $xy = 1$, which means $x \in \bar{A}^\times$; and claim follows. ∎

Lemma. A: Artinian $\Rightarrow$ $\operatorname{Spec} A$ is finite and discrete.

Pf. $\forall\, \mathfrak{p} \in \operatorname{Spec} A$, $V(\mathfrak{p}) = \{\mathfrak{p}\}$ as $\operatorname{Spec} A = \operatorname{Max} A$; and so any point is closed. So if $|\operatorname{Spec} A| < \infty$, then any point is open as well; and this implies $\operatorname{Spec} A$ is discrete.

- Let $\Sigma := \{\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n \mid \mathfrak{m}_i \in \operatorname{Spec} A,\ n \in \mathbb{Z}^+\}$. Since A is Artinian, Σ has a minimal element $\bar{\mathfrak{m}}_1 \cap \cdots \cap \bar{\mathfrak{m}}_k$. So for any $\mathfrak{m} \in \operatorname{Spec} A$, $\mathfrak{m} \cap \bar{\mathfrak{m}}_1 \cap \cdots \cap \bar{\mathfrak{m}}_k = \bar{\mathfrak{m}}_1 \cap \cdots \cap \bar{\mathfrak{m}}_k$. Hence $\bigcap_{i=1}^{k} \bar{\mathfrak{m}}_i \subseteq \mathfrak{m}$, which implies $\exists\, i$, $\bar{\mathfrak{m}}_i \subseteq \mathfrak{m}$. Since $\bar{\mathfrak{m}}_i \in \operatorname{Max} A$, $\bar{\mathfrak{m}}_i = \mathfrak{m}$; and so $\operatorname{Spec} A = \{\bar{\mathfrak{m}}_1, \ldots, \bar{\mathfrak{m}}_k\}$. ∎

Lecture 22: Jacobson radical of an Artinian ring is nilpotent

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Cor. A: Artinian $\Rightarrow$ $J(A) = \mathrm{Nil}(A)$.

Pf. $J(A) = \bigcap_{\mathfrak{m} \in \mathrm{Max}\, A} \mathfrak{m} = \bigcap_{\mathfrak{p} \in \mathrm{Spec}\, A} \mathfrak{p} = \mathrm{Nil}(A)$. ■

Proposition. A: Artinian $\Rightarrow$ $J(A)$ is nilpotent; that means

$J(A)^n = 0$ for some $n \in \mathbb{Z}^+$.

Pf. Suppose to the contrary that $\forall n \in \mathbb{Z}^+$, $J(A)^n \neq 0$. Consider

$J(A) \supseteq J(A)^2 \supseteq \cdots$. Since A is Artinian, $\exists n_0 \in \mathbb{Z}^+$ s.t.

$$0 \neq J_0 := J(A)^{n_0} = J(A)^{n_0+1} = \cdots .$$

Let $\Sigma := \{\mathfrak{a} \trianglelefteq A \mid \mathfrak{a} J_0 \neq 0\}$. Since A is Artinian and $A \in \Sigma$,

Σ has a minimal element $\mathfrak{a}_0$. Since $\mathfrak{a}_0 J_0 \neq 0$, $\exists\, x \in \mathfrak{a}_0$ s.t.

(I) $x J_0 \neq 0$. Therefore $\langle x \rangle \in \Sigma$ and $\langle x \rangle \subseteq \mathfrak{a}_0$; and so

$\mathfrak{a}_0 = \langle x \rangle$. In fact, $x J_0 J_0 = x J_0^2 = x J_0 \neq 0$; and so

$x J_0 \in \Sigma$ and $x J_0 \subseteq \langle x \rangle = \mathfrak{a}_0$, which implies

$$J_0 \langle x \rangle = \langle x \rangle .$$

Since $J_0 \subseteq J(A)$, by Nakayama's lemma $\langle x \rangle = 0$ which

contradicts (I). ■

Lecture 22: Artinian rings are Noetherian

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<u>Theorem</u>. A: Artinian $\Leftrightarrow$ A: Noetherian and $\dim A = 0$.

<u>Pf</u>. ($\Leftarrow$) we have already proved.

($\Rightarrow$) We know ① $\dim A = 0$ ② $|\operatorname{Spec} A| < \infty$ ③ $J(A)^n = 0$ for some $n \in \mathbb{Z}^+$.

$J(A) = \bigcap_{\mathfrak{m} \in \operatorname{Max} A} \mathfrak{m} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_k$ as $|\operatorname{Max} A| < \infty$.

So $\underbrace{\prod_{i=1}^{k} \mathfrak{m}_i^n = 0}_{(\mathrm{I})}$; By a result that we proved in the previous lecture (I) and A: Artinian imply A is Noetherian. ∎

<u>Theorem</u> Any Artinian ring is a unique (up to isomor.) finite direct product of local Artinian rings.

<u>Pf</u>. (Existence). A: Artinian $\Rightarrow$ $J(A)^n = 0$ $\Rightarrow$ $\prod_{i=1}^{k} \mathfrak{m}_i^n = 0$.

Since $\mathfrak{m}_i$ and $\mathfrak{m}_j$ are coprime for $i \neq j$, $\mathfrak{m}_i^n$ and $\mathfrak{m}_j^n$ are coprime ($\sqrt{\mathfrak{m}_i^n + \mathfrak{m}_j^n} \supseteq \mathfrak{m}_i + \mathfrak{m}_j = A$.). Hence by the Chinese Remainder Theorem: (1) $\bigcap_{i=1}^{k} \mathfrak{m}_i^n = \prod_{i=1}^{k} \mathfrak{m}_i^n = 0$

(2) $A \simeq \bigoplus_{i=1}^{k} A/\mathfrak{m}_i^n$. Notice that $A/\mathfrak{m}_i^n$ is Artinian, $\operatorname{Spec}(A/\mathfrak{m}_i^n) \longleftrightarrow V(\mathfrak{m}_i^n) = \{\mathfrak{m}_i\}$; and so $A/\mathfrak{m}_i^n$ is a local ring and $\operatorname{Spec}(A/\mathfrak{m}_i^n) = \{\mathfrak{m}_i/\mathfrak{m}_i^n\}$.

Lecture 22: Structure of Artinian rings

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(Uniqueness) Suppose A_i is Artinian ring and $\operatorname{Max} A_i = \{\mathfrak{m}_i\}$, and $A = A_1 \times \cdots \times A_k$. Let

$$\tilde{\mathfrak{m}}_i := A_1 \times \cdots \times A_{i-1} \times \mathfrak{m}_i \times A_{i+1} \times \cdots \times A_k$$
$$\cup\!|$$
$$\mathfrak{q}_i := A_1 \times \cdots \times A_{i-1} \times \{0\} \times A_{i+1} \times \cdots \times A_k .$$

Hence $\tilde{\mathfrak{m}}_i \in \operatorname{Max} A$ and $\mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_k = 0$.

Since $\mathfrak{m}_i = J(A_i)$, $\mathfrak{m}_i^{n_i} = 0$ for some $n_i \in \mathbb{Z}^+$. Hence $\tilde{\mathfrak{m}}_i^{n_i} = \mathfrak{q}_i$, which implies $\mathfrak{q}_i$ is an $\tilde{\mathfrak{m}}_i$-primary. (as $\tilde{\mathfrak{m}}_i \in \operatorname{Max} A$. So $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_k = 0$ is a primary decomposition, and it is easy to see that it is a reduced primary decomp.

Since $\operatorname{Max} A = \operatorname{Spec} A$, all the elements of $\operatorname{ass}(0)$ are minimal, and so by the 2^{nd} uniqueness theorem, $\mathfrak{q}_i$'s are unique; and claim follows. ∎

Next we focus on local Artinian rings. (Next lecture!)