Lecture 23: Local Artinian rings

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In the previous lecture we proved that any Artinian ring is a unique product of local Artinian rings. Here is on local Artinian rings.

Proposition. A: local Artinian ring, $\operatorname{Max} A = \{\mathfrak{m}\}$. Then TFAE

(1) Any ideal of A is principal (2) $\mathfrak{m}$ is principal (3) $\dim_{A/\mathfrak{m}}\left(\frac{\mathfrak{m}}{\mathfrak{m}^2}\right) \leq 1$.

Pf. Clearly (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1). If $\dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = 0$, then $\mathfrak{m} = \mathfrak{m}^2$.

Since A is Artinian, A is Noetherian; and so $\mathfrak{m}$ is f.g. Hence, by Nakayama's lemma and $J(A) = \mathfrak{m}$, we have $\mathfrak{m} = 0$; this implies A is a field.

If $\dim_{A/\mathfrak{m}}\left(\frac{\mathfrak{m}}{\mathfrak{m}^2}\right) = 1$, then $\exists\, x \in \mathfrak{m} \setminus \mathfrak{m}^2$ s.t. $\langle x \rangle + \mathfrak{m}^2 = \mathfrak{m}$.

Hence again by Nakayama's lemma $\langle x \rangle = \mathfrak{m}$.

Suppose $0 \neq \mathfrak{a} \underset{\neq}{\trianglelefteq} A$. Since A is Artinian, $J(A)^n = 0$; and so $\mathfrak{m}^n = 0$.

$\Rightarrow \exists\, k \in \mathbb{Z}^+$, $\mathfrak{m}^k \supseteq \mathfrak{a}$ and $\mathfrak{m}^{k+1} \not\supseteq \mathfrak{a}$. Suppose $a \in \mathfrak{a} \setminus \mathfrak{m}^{k+1}$.

Then $\exists\, y \in A$ s.t. $a = y x^k$ and $y \in A \setminus \mathfrak{m}$. Hence $x^k \in \mathfrak{a}$, and so $\mathfrak{m}^k = \langle x^k \rangle \subseteq \mathfrak{a} \subseteq \mathfrak{m}^k$; this implies $\mathfrak{a} = \langle x^k \rangle$. ∎

Lecture 23: Dimension 1 Noetherian domains

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Next we study Noetherian rings of dim. 1. For a minimal prime $\mathfrak{p} \in \operatorname{Spec} A$, $A/\mathfrak{p}$ is again Noeth. and of dim. 1; moreover it is an integral domain. And knowing structure of $A/\mathfrak{p}$ tells us a lot about A. So we assume in addition that A is an integral domain. Let $\bar{A}$ be the integral closure of A in its field of fractions. Since $\bar{A}/A$ is an integral extension, we deduce that $\bar{A}$: Noetherian; $\dim \bar{A} = \dim A = 1$;

$\bar{A}$: integral domain; $\bar{A}$: integrally closed.

So we shall focus on understanding:

integral domain, integrally closed, Noetherian, $\dim = 1$.

Such a ring is called a <u>Dedekind domain</u>; and we have seen that the integral closure $\mathcal{O}_k$ of $\mathbb{Z}$ in a finite extension k of $\mathbb{Q}$ is a Dedekind domain.

In order to deal with one prime at a time, we localized A at $\mathfrak{m} \in \operatorname{Max} A = \operatorname{Spec} A \setminus \{0\}$.

Lecture 23: Local Noetherian domain of dim 1

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Since A is integrally closed, $\forall \mathfrak{m} \in \operatorname{Max} A$, $A_{\mathfrak{m}}$ is integrally closed.

Lemma. D: integral domain, local, Noetherian, and $\dim D = 1$. Then

(0) $\operatorname{Spec} D = \{\underbrace{0}_{\text{generic}}, \mathfrak{m}\}$, $\underbrace{\mathfrak{m}}_{\text{special point}}$ is a closed point and 0 is dense.

(1) $0 \neq \mathfrak{a} \underset{\neq}{\lhd} D \Rightarrow \mathfrak{a}$ is $\mathfrak{m}$-primary and $\exists k \in \mathbb{Z}^+$, $\mathfrak{m}^k \subseteq \mathfrak{a}$.

(2) $\mathfrak{m} \supsetneq \mathfrak{m}^2 \supsetneq \cdots$ and $\bigcap_{i=1}^{\infty} \mathfrak{m}^i = 0$.

Pf. (0) is clear. (1) Since $\mathfrak{a} \neq 0$, $V(\mathfrak{a}) \subseteq \operatorname{Spec} D \setminus \{0\} = \{\mathfrak{m}\}$

$\Rightarrow V(\mathfrak{a}) = \{\mathfrak{m}\} \Rightarrow \sqrt{\mathfrak{a}} = \mathfrak{m} \Rightarrow \mathfrak{m}^k \subseteq \mathfrak{a}$ as $\mathfrak{m}$ is f.g.
(D is Noetherian.)

(2) If $\mathfrak{m}^{\ell} = \mathfrak{m}^{\ell+1}$, then $\mathfrak{m}^{\ell} = \underset{(*)}{J(D)\,\mathfrak{m}^{\ell}}$. Since D is Noeth.

$\mathfrak{m}^{\ell}$ is f.g.; and so by Nakayama's lemma and (*) $\mathfrak{m}^{\ell} = 0$.

Since D is an integral domain, $\mathfrak{m} = 0$ which is a contradiction.

If $\bigcap_{i=1}^{\infty} \mathfrak{m}^i \neq 0$, then by part (1) $\mathfrak{m}^k \subseteq \bigcap_{i=1}^{\infty} \mathfrak{m}^i$; this implies $\mathfrak{m}^k = \mathfrak{m}^{k+1}$ which is a contradiction. ∎

Lecture 23: Discrete Valuation Rings

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Thm. D: Noetherian, integral domain, local, $\dim D = 1$; $\operatorname{Max} D = \{\mathfrak{m}\}$.

F: field of fractions of D. TFAE:

(1) D is integrally closed   (2) $\mathfrak{m}$ is principal

(3) $\dim_{k(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2 = 1$ where $D/\mathfrak{m}$.

(4) $0 \neq \mathfrak{a} \underset{\neq}{\trianglelefteq} D \Rightarrow \mathfrak{a} = \mathfrak{m}^k$ for some $k \in \mathbb{Z}^+$

(5) $\exists\, \pi \in D$, $\forall\, 0 \neq \mathfrak{a} \underset{\neq}{\trianglelefteq} D$, $\mathfrak{a} = \langle \pi^k \rangle$ for some $k \in \mathbb{Z}^+$

(6) $\exists\, v : F \to \mathbb{Z} \cup \{\infty\}$, $v(a) = \infty \Leftrightarrow a = 0$

(a valuation) $v(a_1 a_2) = v(a_1) + v(a_2)$

$v(a_1 + a_2) \geq \min\{v(a_1), v(a_2)\}$

and $D = \{a \in F \mid v(a) \geq 0\}$.

[Remark. Because of (6), these rings are called discrete valuation rings.]

Pf. (1) $\Rightarrow$ (2) Suppose $0 \neq a \in \mathfrak{m}$. Then $\exists\, k$, $\langle a \rangle \supseteq \mathfrak{m}^k$. Suppose k is the smallest such integer. So $\langle a \rangle \not\supseteq \mathfrak{m}^{k-1}$, $\langle a \rangle \supseteq \mathfrak{m}^k$. Let be $\mathfrak{m}^{k-1} \setminus \mathfrak{m}^k$. So $b\,\mathfrak{m} \subseteq \langle a \rangle$; let $\alpha := \frac{b}{a} \in F$. Then $\alpha\,\mathfrak{m} \subseteq D$. If $\alpha\,\mathfrak{m} = D$, then $\mathfrak{m} = D\alpha^{-1}$; and we

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we are done.

- If $\alpha\mathfrak{m} \neq D$, then $\alpha\mathfrak{m} \subseteq \mathfrak{m}$ as $\alpha\mathfrak{m}$ is a D-submod of D and $\operatorname{Max} D = \{\mathfrak{m}\}$.

  Since $\mathfrak{m}$ is a f.g. A-mod, $\exists\, a_i \in A$ s.t.

  $$\alpha^n + a_{n-1}\alpha^{n-1} + \dots + a_0 \in \operatorname{Ann} \mathfrak{m} = 0.$$

As D is integrally closed, $\alpha \in D$ which is a contradiction.

(2) $\Rightarrow$ (3) clear.

(3) $\Rightarrow$ (4) $0 \neq \mathfrak{a} \underset{\neq}{\trianglelefteq} D$. Then by lemma $\mathfrak{a} \supseteq \mathfrak{m}^k$ for some k.

Then $\overline{D} := D/\mathfrak{m}^k$ is a zero dimensional, local Noetherian ring, with $\dim_{\overline{D}/\overline{\mathfrak{m}}} \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 = 1$ where $\overline{\mathfrak{m}} := \mathfrak{m}/\mathfrak{m}^k$. So $\overline{D}$ is a local Artinian with $\dim_{\overline{D}/\overline{\mathfrak{m}}} \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 = 1$, which implies $\mathfrak{a}/\mathfrak{m}^k = \mathfrak{m}^i/\mathfrak{m}^k$ for some i; and claim follows.

(4) $\Rightarrow$ (5) Let $\pi \in \mathfrak{m} \setminus \mathfrak{m}^2$. Then by assumption $\exists k$, $\langle \pi \rangle = \mathfrak{m}^k$.

Since $\pi \in \mathfrak{m} \setminus \mathfrak{m}^2$, $k=1$. And so $\mathfrak{m}^k = \langle \pi^k \rangle$.

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(5) $\Rightarrow$ (6) By the discussion prior to the proof of (1) $\Rightarrow$ (2), we have $D \supsetneq \langle \pi \rangle \supsetneq \langle \pi^2 \rangle \supseteq \cdots$. So for any $d \in D$, $\exists! \, v(d) \in \mathbb{Z}^{\geq 0}$ s.t. $\langle d \rangle = \langle \pi^{v(d)} \rangle$.

Let $v(a/b) := v(a) - v(b)$; one can rather easily check that v is a (discrete) valuation of F.

(6) $\Rightarrow$ (1) D is a valuation ring; and so it is integrally closed. ∎