Lecture 24: DVR

Monday, May 28, 2018 10:22 PM

<u>Recall</u>. In the previous lecture we proved the following important theorem:

<u>Theorem</u>. A: Integral domain, Noetherian, $\dim A = 1$, $\mathrm{Max}\, A = \{\mathfrak{m}\}$.

TFAE: (1) A is integrally closed (2) $\mathfrak{m}$ is principal

(3) $\dim_{k(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2 = 1$ where $k(\mathfrak{m}) := A/\mathfrak{m}$.

(4) For any $0 \neq \mathfrak{a} \underset{\neq}{\trianglelefteq} A$, $\exists i$, $\mathfrak{a} = \mathfrak{m}^i$.

(5) $\exists \pi$ s.t. $\forall 0 \neq \mathfrak{a} \underset{\neq}{\trianglelefteq} A$, $\exists i$, $\mathfrak{a} = \langle \pi^i \rangle$.

(6) $\exists v: F \to \mathbb{Z} \cup \{\infty\}$ s.t. $\cdot\, v(\alpha) = \infty \Leftrightarrow \alpha = 0$

(Discrete Valuation Ring) $\cdot\, v(\alpha_1 \alpha_2) = v(\alpha_1) + v(\alpha_2)$

(F: field of frac. of A) $\cdot\, v(\alpha_1 + \alpha_2) \geq \min\{v(\alpha_1), v(\alpha_2)\}$

$\cdot\, a \in A \Leftrightarrow v(a) \geq 0$.

Next we will see the global analogue of this statement.

<u>Theorem</u>. A: integral domain, Noetherian, $\dim A = 1$. TFAE:

(1) A: integrally closed (2) $A_{\mathfrak{m}}$: DVR, $\forall \mathfrak{m} \in \mathrm{Max}\, A$

(3) $\mathfrak{q} \trianglelefteq A$ primary $\Leftrightarrow \mathfrak{q} = \mathfrak{p}^n$ for some $\mathfrak{p} \in \mathrm{Spec}\, A$.

Lecture 24: Dedekind domains

Friday, May 25, 2018 8:54 AM

<u>Def.</u> A ring that satisfies the above properties is called a <u>Dedekind Domain</u>.

<u>Cor.</u> Suppose $\mathcal{O}_k$ is the ring of integers of a number field. Then $\mathcal{O}_k$ is a Dedekind domain; and so $\forall 0 \neq \mathfrak{a} \underset{\neq}{\trianglelefteq} D$,

$$\mathfrak{a} = \prod_{\mathfrak{m} \in \operatorname{Max} D} \mathfrak{m}^{v_{\mathfrak{m}}(\mathfrak{a})}$$

and $v_{\mathfrak{m}}(\mathfrak{a}) = 0$ except for finitely many $\mathfrak{m}$.

<u>Pf.</u> . We have already proved that $\mathcal{O}_k \simeq \mathbb{Z}^{[k:\mathbb{Q}]}$ as an abelian group and in particular it is Noetherian;

. $\mathcal{O}_k$ is the integral closure of $\mathbb{Z}$ in k; hence it is integrally closed; and $\dim \mathcal{O}_k = \dim \mathbb{Z} = 1$.

. By the 2^{nd} uniqueness theorem, $\mathfrak{a}$ has a unique reduced primary decomposition. Since $\mathcal{O}_k$ is Dedekind, any primary ideal is a power of a prime ideal. Since $\mathfrak{a} \neq 0$, $\operatorname{Ass}(\mathfrak{a}) \subseteq \operatorname{Max}(\mathcal{O}_k)$. For $\mathfrak{m} \in \operatorname{Max}(\mathcal{O}_k)$, $\mathfrak{m} \underset{\neq}{\supset} \mathfrak{m}^2 \underset{\neq}{\supset} \cdots$; and so by the Chinese Remainder Theorem claim follows. ■

Lecture 24: Dedekind domain

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Pf of Theorem. • A: integrally closed $\Rightarrow \forall \mathfrak{m} \in \operatorname{Max} A$, $A_{\mathfrak{m}}$ is integrally closed.

A: Noeth., $\dim A = 1 \Rightarrow \dim A_{\mathfrak{m}} = 1$ and $A_{\mathfrak{m}}$ Noeth. Hence

$A_{\mathfrak{m}}$ is a DVR.

• $\mathfrak{q} \neq 0$ primary $\Rightarrow \sqrt{\mathfrak{q}} = \mathfrak{m} \in \operatorname{Max} A$ as $\dim A = 1$; and

$\mathfrak{q}_{\mathfrak{m}}$ is $\mathfrak{m}A_{\mathfrak{m}}$-primary $\Rightarrow \mathfrak{q}_{\mathfrak{m}} = \mathfrak{m}^n A_{\mathfrak{m}}$ for some $n \in \mathbb{Z}^+$

as $A_{\mathfrak{m}}$ is a DVR. Since $\mathfrak{m}^n$ is $\mathfrak{m}$-primary and $\mathfrak{q}_{\mathfrak{m}} = \mathfrak{m}^n_{\mathfrak{m}}$,

$\mathfrak{q} = \mathfrak{m}^n$.

• Any non-zero ideal $\tilde{\mathfrak{a}}$ of $A_{\mathfrak{m}}$ is $\mathfrak{m}A_{\mathfrak{m}}$-primary; and so $\exists \mathfrak{q} \trianglelefteq A$

$\mathfrak{q}$: $\mathfrak{m}$-primary and $\tilde{\mathfrak{a}} = \mathfrak{q}_{\mathfrak{m}}$. By assumption $\mathfrak{q} = \mathfrak{m}^n$; hence

$\tilde{\mathfrak{a}} = (\mathfrak{m}A_{\mathfrak{m}})^n$. Therefore $A_{\mathfrak{m}}$ is a DVR. This implies

$A_{\mathfrak{m}}$ is integrally closed for any $\mathfrak{m} \in \operatorname{Max} A$. Hence A is

integrally closed. ∎

As we have seen before $\mathcal{O}_k$ is not necessarily a PID. Next

we want to have a way of saying how "badly" $\mathcal{O}_k$ is faily of

being a PID.

Lecture 24: Class group

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Def. A: integral domain; F: field of fractions;

$$\mathrm{Frac}(A) := \left\{ M \subseteq F \;\middle|\; \begin{array}{l} M: A\text{-submod};\ M \neq 0; \\ \exists \alpha \in F^\times,\ \alpha M \subseteq A \end{array} \right\};$$

$$\mathrm{Prin}(A) := \{\alpha A \mid \alpha \in F^\times\}.$$

Lemma. • $M_1, M_2 \in \mathrm{Frac}(A) \Rightarrow M_1 M_2 \in \mathrm{Frac}(A)$,

where $M_1 M_2 := \sum_{m_i \in M_i} A\, m_1 m_2$.

• $M \in \mathrm{Frac}(A) \Rightarrow M \cdot A = A \cdot M = M$.

• $(\mathrm{Prin}(A), \cdot)$ is a group.

Pf. Clear. ∎

Lemma. For $M \in \mathrm{Frac}(A)$, $(A:M) := \{\alpha \in F \mid \alpha M \subseteq A\} \in \mathrm{Frac}(A)$, and M has an inverse in $\mathrm{Frac}(A)$ if and only if $(A:M)M = A$.

Pf. $0 \neq (A:M)$ is a submodule of F and, for $\beta \in M \setminus \{0\}$, $\beta (A:M) \subseteq A$; and so $(A:M) \in \mathrm{Frac}(A)$.

. If $(A:M)M = A$, then M is invertible in $\mathrm{Frac}(A)$ by definition.

. If $M'M = A$ for some $M' \in \mathrm{Frac}(A)$, then $M' \subseteq (A:M)$; and so $A \subseteq (A:M)M \subseteq A$; and claim follows. ∎

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<u>Proposition</u>. TFAE. (1) $M \in \mathrm{Frac}(A)$ is invertible.

(2) M is f.g. and $\forall \mathfrak{p} \in \mathrm{Spec}(A)$, $M_{\mathfrak{p}} \in \mathrm{Frac}(A_{\mathfrak{p}})$ is invertible.

(3) M is f.g. and $\forall \mathfrak{m} \in \mathrm{Max}(A)$, $M_{\mathfrak{m}} \in \mathrm{Frac}(A_{\mathfrak{m}})$ is invertible.

<u>Pf</u>. (1) $\Rightarrow$ (2), $MM' = A$ implies $\exists\, m_i \in M$, $m_i' \in M'$ s.t. $\sum_{i=1}^{k} m_i m_i' = 1$.

Then, for any $x \in M$, $x = x \cdot 1 = \sum_{i=1}^{n} \underbrace{(x m_i')}_{\text{in } A} m_i \in \langle m_1, \ldots, m_k \rangle$.

And so $M = \langle m_1, \ldots, m_k \rangle$ is a f.g. A-mod.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1) To show $(A:M)M = A$, it is enough to show for any $\mathfrak{m} \in \mathrm{Max}\, A$, $((A:M)M)_{\mathfrak{m}} = A_{\mathfrak{m}}$. It is clear that $((A:M)M)_{\mathfrak{m}} = (A:M)_{\mathfrak{m}} M_{\mathfrak{m}}$. We also have

$$(A_{\mathfrak{m}} : M_{\mathfrak{m}}) = \{\alpha \in F \mid \alpha M_{\mathfrak{m}} \subseteq A_{\mathfrak{m}}\} = \{\alpha \in F \mid \alpha M \subseteq A_{\mathfrak{m}}\}$$

$$= \{\alpha \in F \mid \alpha x_1, \ldots, \alpha x_k \in A_{\mathfrak{m}}\} \quad \text{where } M = \langle x_1, \ldots, x_k \rangle$$

$$= \{\alpha \in F \mid \alpha x_1 = \tfrac{a_1}{s_1}, \ldots, \alpha x_k = \tfrac{a_k}{s_k} \text{ for some } a_i \in A,\ s_i \in A \setminus \mathfrak{m}\}$$

$$= \{\alpha \in F \mid \exists s \in A \setminus \mathfrak{m},\ s\alpha x_i \in A\} \quad (s = s_1 \cdots s_k)$$

$$= (A:M)_{\mathfrak{m}};$$ and claim follows. ∎

Lecture 24: Class group

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Corollary. A: Dedekind domain $\Rightarrow$ all elements of $\mathrm{Frac}(A)$ are invertible.

Pf. • A: Dedekind domain $\Rightarrow$ $\forall \mathfrak{m} \in \mathrm{Max}\, A$, $A_{\mathfrak{m}}$ is a DVR.

and, if $\alpha M \subseteq A$, implies αM is f.g. as A is Noeth.

And so M is f.g.

• Let N be a f.g. $A_{\mathfrak{m}}$-submod of F. Suppose

$N = A_{\mathfrak{m}}\alpha_1 + \cdots + A_{\mathfrak{m}}\alpha_k$ and $\alpha_i = u_i \pi^{n_i}$, $v(u_i) = 0$,

$v(\pi) = 1$. Then $N = A_{\mathfrak{m}} \pi^{\min(n_1, \ldots, n_k)}$; and so

$(A_{\mathfrak{m}} : N) = \pi^{-\min(n_1, \ldots, n_k)} A_{\mathfrak{m}}$ which implies

$(A_{\mathfrak{m}} : N)\, N = A_{\mathfrak{m}}$. ∎

Def. The class group of A is $\mathrm{Cl}(A) := \mathrm{Frac}(A)/\mathrm{Prin}(A)$.

Cor. $\mathrm{Cl}(A) = 0 \Leftrightarrow A$ is a PID.