Lecture 26: Krull's height theorem

Sunday, June 3, 2018 7:15 PM

Thm (Krull's height theorem) Suppose A is Noetherian, $\mathfrak{a}=\langle a_1,\dots,a_n\rangle\neq A$, $\mathfrak{p}$ is minimal in $V(\mathfrak{a})$. Then $ht(\mathfrak{p})\le n$.

<u>Pf.</u> We proceed by induction on n. The base of induction follows from Krull's Principal Ideal Theorem.

Suppose $\mathfrak{p}\in Ass(\mathfrak{a})$ is a minimal element; and we have to show $ht(\mathfrak{p})\le n$. Notice that $ht(\mathfrak{p})=ht(\mathfrak{p}A_{\mathfrak{p}})$ and $\mathfrak{p}A_{\mathfrak{p}}$ is a minimal element of $\mathfrak{p}A_{\mathfrak{p}}\in Ass(\mathfrak{a}_{\mathfrak{p}})$. So w.l.o.g. we can and will assume A is a local ring and $\{\mathfrak{p}\}=Max\,A$. Therefore $V(\mathfrak{a})=\{\mathfrak{p}\}$ (*) as $\mathfrak{p}$ is both minimal and the only maximal ideal.

Suppose $\mathfrak{p}'\subsetneq\mathfrak{p}$ is a prime ideal and there is no prime ideal between $\mathfrak{p}'$ and $\mathfrak{p}$. By (*), $\exists\, i$, $a_i\notin\mathfrak{p}'$. W.L.O.G. we can and will assume $a_n\notin\mathfrak{p}'$. Since there is no prime between $\mathfrak{p}'$ and $\mathfrak{p}$, $V(\langle a_n\rangle+\mathfrak{p}')=\mathfrak{p}$. And so $\sqrt{\langle a_n\rangle+\mathfrak{p}'}=\mathfrak{p}$

Hence, for any i, $\exists\, m_i\in\mathbb{Z}^+$, $a_i^{m_i}=a_n r_i+b_i$ for some $r_i\in A$

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Friday, June 1, 2018 8:28 AM

$b_i \in \mathfrak{p}'$.

<u>Claim</u>. $V(\langle a_n, b_1, \ldots, b_{n-1}\rangle) = \{\mathfrak{p}\}$.

<u>Pf</u>. $\langle a_n, b_1, \ldots, b_{n-1}\rangle \ni a_i^{m_i}$ for $1 \le i \le n-1 \Rightarrow a_i \in \sqrt{\langle a_n, b_1, \ldots, b_{n-1}\rangle}$

$\Rightarrow \mathfrak{p} = \sqrt{\langle a_1, \ldots, a_n\rangle} \subseteq \sqrt{\langle a_n, b_1, \ldots, b_{n-1}\rangle} \subseteq \mathfrak{p}$. $\square$

<u>Claim</u>. $\mathfrak{p}'$ is a minimal element of $V(\langle b_1, \ldots, b_{n-1}\rangle)$.

<u>Pf</u>. Let $\overline{A} := A/\langle b_1, \ldots, b_{n-1}\rangle$, $\overline{\mathfrak{p}}' := \mathfrak{p}'/\langle b_1, \ldots, b_{n-1}\rangle$, and $\overline{\mathfrak{p}} := \mathfrak{p}/\langle b_1, \ldots, b_{n-1}\rangle$. By the previous claim, $\overline{\mathfrak{p}}$ is a minimal element of $V(\langle \bar{a}_1\rangle)$. Hence by Krull's PIT, $\mathrm{ht}(\overline{\mathfrak{p}}) \le 1$. (*)

Since $\overline{\mathfrak{p}}' \in \mathrm{Spec}(\overline{A})$ and $\overline{\mathfrak{p}}' \subsetneq \overline{\mathfrak{p}}$, (*) implies that $\overline{\mathfrak{p}}'$ is is a minimal prime; and claim follows. $\square$

By the above claim and the induction hypothesis, $\mathrm{ht}(\mathfrak{p}') \le n-1$.

Since this is true for any such $\mathfrak{p}'$, $\mathrm{ht}(\mathfrak{p}) \le (n-1)+1 = n$; and claim follows. $\blacksquare$

<u>Corollary</u>. Suppose k is a field. Then $\dim k[x_1, \ldots, x_n] = n$.

<u>Pf</u>. Let $\bar{k}$ be an algebraic closure of k. Then $\bar{k}[x_1, \ldots, x_n]$ is

integral over $k[x_1,\dots,x_n]$. Hence $\dim k[x_1,\dots,x_n] = \dim \bar{k}[x_1,\dots,x_n]$.

We have $\dim \bar{k}[x_1,\dots,x_n] = \sup\{ ht(\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{Max}(\bar{k}[x_1,\dots,x_n])\}$

$\boxed{\text{Hilbert's Nullstellensatz}}$ $\;\; = \sup\{ ht(\langle x_1-p_1,\dots,x_n-p_n\rangle) \mid (p_1,\dots,p_n)\in \bar{k}^n\}$

$\boxed{\text{Krull's height theorem}}$ $\;\; \leq n \qquad$ (I)

On the other hand $0 \subsetneq \langle x_1\rangle \subsetneq \langle x_1,x_2\rangle \subsetneq \cdots \subsetneq \langle x_1,\dots,x_n\rangle$ is a chain of length n of prime ideals. Hence $\dim \bar{k}[x_1,\dots,x_n] \geq n$ (II)

(I), (II) imply the claim. ■

<u>Remark.</u> The above proof implies more:

$$\forall\, \mathfrak{m}\in \mathrm{Max}(k[x_1,\dots,x_n]), \quad ht(\mathfrak{m}) = n.$$

<u>Pf.</u> Since $\bar{k}[x_1,\dots,x_n] / k[x_1,\dots,x_n]$ is integral,

$f^*: \mathrm{Spec}\,\bar{k}[x_1,\dots,x_n] \to \mathrm{Spec}\, k[x_1,\dots,x_n]$ is an onto finite (open closed) map. And f^* induces a bij. between maximal ideals. Suppose $\tilde{\mathfrak{m}} \in \mathrm{Max}\,\bar{k}[x_1,\dots,x_n]$ s.t. $f^*(\tilde{\mathfrak{m}}) = \mathfrak{m}$. Then $ht(\mathfrak{m}) = ht(\tilde{\mathfrak{m}})$ and by Hilbert's Nullstellensatz and Krull's HT as above we get $ht(\tilde{\mathfrak{m}}) = n$. ■

Lecture 26: Proof of Krull's PIT

Monday, June 4, 2018 8:03 AM

Pf of Krull's PIT. Suppose $\mathfrak{p}$ is a minimal element of $V(\langle a\rangle)$.

Since $ht(\mathfrak{p}) = ht(\mathfrak{p}A_{\mathfrak{p}})$ and $\mathfrak{p}A_{\mathfrak{p}}$ is a minimal element of $V(\langle a\rangle)$, w.l.o.g. we can and will assume A is a local ring and $\mathrm{Max}\, A = \{\mathfrak{p}\}$.

Since $\mathfrak{p}$ is minimal in $V(\langle a\rangle)$ and $\mathfrak{p}$ is the only maximal ideal, $V(\langle a\rangle) = \{\mathfrak{p}\}$; and so $\mathrm{spec}(A/\langle a\rangle) = \{\mathfrak{p}\}$ which implies $\dim A/\langle a\rangle = 0$; thus $A/\langle a\rangle$ is a local Artinian ring.

Suppose to the contrary that $ht(\mathfrak{p}) \geq 2$; and so $\exists\, \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}$ s.t. $\mathfrak{p}_i \in \mathrm{Spec}\, A$. Going to $A/\mathfrak{p}_0$, we still have a local Noetherian ring of dim ≥ 2; and $V(\langle a\rangle + \mathfrak{p}_0/\mathfrak{p}_0) = \{\mathfrak{p}\}$.

So w.l.o.g. we can and will assume A is an integral domain.

We would like to say $ht(\mathfrak{p}_1) = 0$, which is equivalent to showing $\dim A_{\mathfrak{p}_1} = 0$. This happens precisely when $\mathfrak{p}_1^n A_{\mathfrak{p}_1} = \mathfrak{p}_1^{n+1} A_{\mathfrak{p}_1}$ for some n. These are $\mathfrak{p}_1 A_{\mathfrak{p}_1}$-primary; and so we can work with their contractions $\mathfrak{p}_1^{(n)} := \mathfrak{p}_1^n A_{\mathfrak{p}_1} \cap A$

We will continue next time.