Lecture 28: Dimension reduction by one

Monday, June 4, 2018 8:47 AM

Theorem. Suppose A is a local Noetherian ring and $a \notin D(A) \cup A^{\times}$.

Then $\dim A/\langle a \rangle = \dim A - 1$.

Pf Suppose $\operatorname{Max} A = \{\mathfrak{m}\}$. Then $\dim A = \operatorname{ht} \mathfrak{m} < \infty$ by Krull's HT.

Similarly $\dim A/\langle a \rangle = d < \infty$; and let

$$\mathfrak{p}_0/\langle a \rangle \subsetneq \mathfrak{p}_1/\langle a \rangle \subsetneq \cdots \subsetneq \mathfrak{p}_d/\langle a \rangle = \mathfrak{m}/\langle a \rangle$$

be a saturated chain of prime ideals. By the previous theorem

$\exists\, \bar{a}_1, \ldots, \bar{a}_d$ s.t. $\mathfrak{m}/\langle a \rangle$ is a minimal prime of $V(\langle \bar{a}_1, \ldots, \bar{a}_d \rangle)$.

Since $\operatorname{ht}(\mathfrak{p}_0/\langle a \rangle) = 0$, $\mathfrak{p}_0$ is a minimal prime in $V(\langle a \rangle)$.

And so $\operatorname{ht} \mathfrak{p}_0 \le 1$ by Krull's PIT. Since a is not a zero-divisor,

$\operatorname{ht} \mathfrak{p}_0 \neq 0$; and so $\operatorname{ht} \mathfrak{p}_0 = 1$. Let $\mathfrak{p}_{-1} \subsetneq \mathfrak{p}_0$. This implies

$\operatorname{ht} \mathfrak{m} \ge d+1$. As $\mathfrak{m}$ is a minimal prime of $V(\langle a, a_1, \ldots, a_d \rangle)$,

by Krull's HT, $\operatorname{ht} \mathfrak{m} \le d+1$. Hence $\dim A = \operatorname{ht} \mathfrak{m} = d+1$. ∎

Def. Suppose A is a local Noetherian ring and $\operatorname{Max} A = \{\mathfrak{m}\}$.

$(x_1, \ldots, x_r)$ is called an <u>A-regular sequence</u> if $x_i \in \mathfrak{m}$

and for any i, $x_i \notin D(A/\langle x_1, \ldots, x_{i-1} \rangle)$.

Lecture 28: A-regular sequence

Friday, June 8, 2018 2:01 AM

<u>Proposition</u> Suppose A is a local Noetherian ring and $(x_1,\dots,x_r)$ is an A-regular sequence. Then, for any $s\leq r$,

$$\dim\left(A/\langle x_1,\dots,x_s\rangle\right)=\dim A-s.$$

In particular, $r\leq \dim A$.

<u>Pf.</u> We prove this by induction on s.

$$\bar{x}_{s+1}\notin D(A/\langle x_1,\dots,x_s\rangle)\cup (A/\langle x_1,\dots,x_s\rangle)^{\times} \Rightarrow$$

$$\dim\left(A/\langle x_1,\dots,x_{s+1}\rangle\right)=\dim\left(A/\langle x_1,\dots,x_s\rangle\right)-1=\dim A-s-1. \blacksquare$$

<u>Def.</u> <u>Depth</u> of a local (Noetherian) ring A is the maximum length of an A-regular sequence.

<u>Cor.</u> A: local Noeth. $\Rightarrow$ $\operatorname{depth}(A)\leq \dim A$.

Does any local Noetherian ring has an A-regular sequence of length <u>$\dim A$</u>? No

$A:=\left(k[x,y]/\langle x^2,xy\rangle\right)_{\langle\bar{x},\bar{y}\rangle}$. $0=\langle\bar{x},\bar{y}\rangle^2\cap\langle\bar{x}\rangle$, and

$D(A)=\langle\bar{x},\bar{y}\rangle$. $\Rightarrow$ $\operatorname{depth}(A)=0$; and $\dim A=1$ as $\langle\bar{x}\rangle\subsetneq\langle\bar{x},\bar{y}\rangle$.

Lecture 28: Cohen-Macaulay rings

Friday, June 8, 2018 7:52 AM

<u>Def</u>. A local Noetherian ring A is called Cohen-Macaulay if

$$\operatorname{depth}(A) = \dim(A)$$

<u>Theorem</u>. Suppose A is a local Noetherian ring and any ideal $\mathfrak{a} \precneqq A$ is <u>unmixed</u>; that means any $\mathfrak{p} \in \operatorname{Ass}(\mathfrak{a})$ is minimal in $\operatorname{Ass}(\mathfrak{a})$. Then A is CM.

<u>Pf</u>. By induction on r, we show there is an A-regular sequence $(x_1, \ldots, x_r)$ if $r \leq \dim A =: d$.

<u>Base</u>. If $d=0$, we are done. Suppose $d \geq 1$. <u>Claim</u>. $\mathfrak{m} \not\subseteq D(A)$, where $\operatorname{Max} A = \{\mathfrak{m}\}$. If $\mathfrak{m} \subseteq D(A) = \bigcup_{\mathfrak{p} \in \operatorname{Ass}(0)} \mathfrak{p}$, then $\mathfrak{m} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass}(0)$; and so $\mathfrak{m} \in \operatorname{Ass}(0)$; Since 0 is unmixed, $\mathfrak{m}$ is a minimal prime. Hence $\dim A = 0$; this is a contradiction.

<u>Induction step</u> If $r=d$, we are done. Suppose $r<d$. Then $\dim A/\langle x_1, \ldots, x_r\rangle = d-r > 0$ by the previous proposition. Hence as $\langle x_1, \ldots, x_r \rangle$ is unmixed, $\mathfrak{m} \notin \operatorname{Ass}(\langle x_1, \ldots, x_r\rangle)$. Therefore $\mathfrak{m}/\langle x_1, \ldots, x_r\rangle \not\subseteq D(A/\langle x_1, \ldots, x_r\rangle)$, and we can find x_{r+1}. ∎

Lecture 28: Finitely generated algebras

Friday, June 8, 2018 2:41 AM

<u>Remark</u>. We only need to assume any ideal $\mathfrak{a}$ with $d(\mathfrak{a}) \le \dim(A/\mathfrak{a})$ is unmixed. And in fact converse of this statement is correct as well.

<u>Recall</u>. For $\mathfrak{p} \in \operatorname{Spec} A$, $\operatorname{ht} \mathfrak{p} + \dim A/\mathfrak{p} =$ max. length of chain of primes that contain $\mathfrak{p}$

And so $\operatorname{ht} \mathfrak{p} + \dim A/\mathfrak{p} \le \dim A$. Equality does not hold in general.

<u>Ex.</u> Let $\mathfrak{a} = \langle x, y\rangle \cap \langle x-1\rangle \trianglelefteq k[x,y]$, and $A := k[x,y]/\mathfrak{a}$.

Then $\operatorname{Ass}(\bar{0}) = \{\underbrace{\langle \bar{x}, \bar{y}\rangle}_{\mathfrak{p}}, \underbrace{\langle \bar{x}-1\rangle}_{\mathfrak{p}'}\}$. Hence $\operatorname{ht}(\mathfrak{p}) = 0$, $\operatorname{ht}(\mathfrak{p}') = 0$;

$A/\mathfrak{p} \simeq k \Rightarrow \dim A/\mathfrak{p} = 0$. And $A/\mathfrak{p}' \simeq k[x,y]/\langle x-1\rangle \simeq k[y]$

$\Rightarrow \dim A/\mathfrak{p}' = 1$.

Hence $\operatorname{ht} \mathfrak{p} + \dim A/\mathfrak{p} = 0 < \operatorname{ht} \mathfrak{p}' + \dim A/\mathfrak{p}' = 1 = \dim A$.

(For any Noetherian ring A, $\dim A = \max\limits_{\mathfrak{p} \in \operatorname{Ass}(0)} \{\operatorname{ht} \mathfrak{p} + \dim A/\mathfrak{p}\}$ as any maximal chain of primes contains a minimal prime.)

<u>Theorem</u>. Suppose k is a field, and A is a finitely generated k-algebra, integral domain. Then any maximal chain of prime ideals of A has length $\dim A$. In particular, for any $\mathfrak{p} \in \operatorname{Spec} A$,

$$\dim A = \operatorname{ht} \mathfrak{p} + \dim A/\mathfrak{p}.$$ (Next lecture).