Lecture 29: Finitely generated algebras

Friday, June 8, 2018 8:46 AM

As we discussed in the previous lecture, $\dim A \geq \operatorname{ht}\mathfrak{p} + \dim A/\mathfrak{p}$ for any $\mathfrak{p} \in \operatorname{Spec} A$; but equality does not necessarily holds.

<u>Thm.</u> k: field; A: f.g. k-algebra; A: integral domain. Then

$$\forall \mathfrak{p} \in \operatorname{Spec} A, \ \operatorname{ht}\mathfrak{p} + \dim A/\mathfrak{p} = \dim A.$$

We start with a stronger form of Noether's normalization lemma.

<u>Strong version of Noether's normalization lemma</u>

Let $\tilde{A} = k[x_1, \ldots, x_n]$ be the ring of polynomials over a field k.

Suppose $\mathfrak{a} \underset{\neq}{\trianglelefteq} \tilde{A}$. Then $\exists\, y_1, \ldots, y_n \in \tilde{A}$ s.t.

(1) $y_1, \ldots, y_n$ are algebraically indep. over k.

(2) $\tilde{A}$ is integral over $k[y_1, \ldots, y_n]$.

(3) $\mathfrak{a} \cap k[y_1, \ldots, y_n] = \langle y_{d+1}, \ldots, y_n \rangle$.

<u>Remark</u>. Connection with the previous version of Noether's normalization lemma: a f.g. k-algebra $A \simeq k[x_1, \ldots, x_n]/\mathfrak{a}$.

Choose y_i's as above for $\mathfrak{a}$. Then A is integral over

Lecture 29: Noether's normalization lemma revisited

Tuesday, June 12, 2018 10:33 PM

$k[y_1,\ldots,y_n]/\langle y_{d+1},\ldots,y_n\rangle \simeq k[y_1,\ldots,y_d]$.)

Pf. Step 1. $\mathfrak{a}=\langle f\rangle$.

Pf of step 1. We have proved that $\phi_M: k[x_1,\ldots,x_n]\to k[x_1,\ldots,x_n]$

$\phi_M(x_n)=x_n$, $\phi_M(x_i)=x_n^{M^i}+x_i$ for $1\le i\le n-1$ is

an automorphism of $\tilde{A}$ and, if $M\gg 1$, then $\phi_M(f)$ is

monic as a poly. in terms of x_n . So applying ϕ_M ,

w.l.o.g. we can and will assume f is monic in terms

of x_n. Let $y_1=x_1,\ldots,y_{n-1}=x_{n-1}$, and $y_n=f$.

Then x_n is integral over $k[y_1,\ldots,y_n]$ (as f is monic

in terms of x_n); y_i's are algebraically indep. over k ;

and $\langle f\rangle\cap k[y_1,\ldots,y_n]=y_n\, k[y_1,\ldots,y_n]$.

Step 2. We proceed by induction on n .

. Base . If $n=1$, then $k[x_1]$ is a PID $\Rightarrow$ $\mathfrak{a}$ is principal.

. Induction step. For $f\in\mathfrak{a}$, $\exists\, y_1,\ldots,y_n$ as in step 1 .

Consider $\mathfrak{a}':=\mathfrak{a}\cap k[y_1,\ldots,y_{n-1}]\underset{\neq}{\trianglelefteq} k[y_1,\ldots,y_{n-1}]$. By the

Lecture 29: Noether's normalization lemma revisited

Wednesday, June 13, 2018 8:42 AM

induction hypothesis, $\exists\ z_1, \ldots, z_{n-1} \in k[y_1, \ldots, y_{n-1}]$ s.t.

(1) z_i's are alg. inde. over k.

(2) $k[y_1, \ldots, y_{n-1}]$ is integral over $k[z_1, \ldots, z_{n-1}]$

(3) $\mathfrak{a}' \cap k[z_1, \ldots, z_{n-1}] = \langle z_{d+1}, \ldots, z_{n-1} \rangle$.

Consider $k[z_1, \ldots, z_{n-1}, y_n]$:

(1) $\left.\begin{array}{l} y_1, \ldots, y_n \text{ alg. ind.} \\ z_1, \ldots, z_{n-1} \in k[y_1, \ldots, y_{n-1}] \\ \quad \text{alg. ind.} \end{array}\right\} \Rightarrow z_1, \ldots, z_{n-1}, y_n$ are alg. inde.

(2) $\left.\begin{array}{l} k[x_1, \ldots, x_n] / k[y_1, \ldots, y_n] \text{ integ.} \\ k[y_1, \ldots, y_{n-1}] / k[z_1, \ldots, z_{n-1}] \text{ inte.} \end{array}\right\} \Rightarrow k[x_1, \ldots, x_n] / k[z_1, \ldots, z_{n-1}, y_n]$ integ.

(3) $\mathfrak{a} \cap k[z_1, \ldots, z_{n-1}, y_n] \underset{?}{=} \langle z_{d+1}, \ldots, z_{n-1}, y_n \rangle$

($\supseteq$) is clear. ($\subseteq$) $g \in \mathfrak{a} \cap k[z_1, \ldots, z_{n-1}, y_n] \Rightarrow$

$g = g_0 + y_n p$ where $g_0 \in k[z_1, \ldots, z_{n-1}]$, $p \in k[z_1, \ldots, z_{n-1}, y_n]$.

Since $y_n \in \mathfrak{a}$, $g_0 \in \mathfrak{a} \cap k[z_1, \ldots, z_{n-1}] = \langle z_{d+1}, \ldots, z_{n-1} \rangle$.

Hence $g \in \langle z_{d+1}, \ldots, z_{n-1}, y_n \rangle$. ∎

Lecture 29: Finitely generated k-domains

Wednesday, June 13, 2018 8:52 AM

<u>pf of theorem</u>. By Noether's normalization lemma, $\exists\, x_1,\dots,x_d \in A$ s.t.

(1) x_i's are algebraically independent over k.

(2) A is integral over $k[x_1,\dots,x_d]$.

Since $k[x_1,\dots,x_d]$ is a UFD, it is integrally closed. Hence

$f^*: \operatorname{Spec} A \longrightarrow \operatorname{Spec} k[x_1,\dots,x_d]$ has the Going-Up and the

Going-Down properties. Therefore

$$\operatorname{ht}(\mathfrak{p}) = \operatorname{ht}(f^*(\mathfrak{p})) \qquad (\text{Going-Down}) \qquad \text{and}$$

$$\dim(A/\mathfrak{p}) = \dim\left(k[x_1,\dots,x_d]/f^*(\mathfrak{p})\right) \qquad (\text{Going-Up}).$$

By the stronger version of Noether's normalization lemma,

$\exists\, y_1,\dots,y_d \in k[x_1,\dots,x_d]$ s.t. (1) y_i's are algebraically indep. over k.

(2) $k[x_1,\dots,x_d]$ is integral over $k[y_1,\dots,y_d]$.

(3) $g^*(f^*(\mathfrak{p})) = \langle y_{d'+1},\dots,y_d\rangle$ where

$g^*: \operatorname{Spec} k[x_1,\dots,x_d] \to \operatorname{Spec} k[y_{d'+1},\dots,y_d]$ is the contra. map.

Again by Going-Down and Going-Up,

$\operatorname{ht}(\mathfrak{p}) = \operatorname{ht}(f^*(\mathfrak{p})) = \operatorname{ht}(g^* f^*(\mathfrak{p})) = \operatorname{ht}(\langle y_{d'+1},\dots,y_d\rangle)$ and

$$\dim(A/\mathfrak{p}) = \dim\left(k[x_1,\dots,x_d]/f^*(\mathfrak{p})\right) = \dim\left(k[y_1,\dots,y_d]/\langle y_{d'+1},\dots,y_n\rangle\right)$$

Since $k[y_1,\dots,y_d]/\langle y_{d'+1},\dots,y_d\rangle \simeq k[y_1,\dots,y_{d'}]$, we deduce

$$\dim A/\mathfrak{p} = d'. \qquad \text{(I)}$$

By Krull's HT, $\operatorname{ht}(\langle y_{d'+1},\dots,y_d\rangle) \le d-d'$; and we have that

$$0 \subsetneq \langle y_{d'+1}\rangle \subsetneq \langle y_{d'+1}, y_{d'+2}\rangle \subsetneq \cdots \subsetneq \langle y_{d'+1},\dots,y_d\rangle \text{ is a}$$

chain of prime ideals of length $d-d'$. Hence

$$\operatorname{ht}(\mathfrak{p}) = \operatorname{ht}(\langle y_{d'+1},\dots,y_d\rangle) = d-d'. \qquad \text{(II)}$$

(I), (II) $\Rightarrow$ $\operatorname{ht}\mathfrak{p} + \dim A/\mathfrak{p} = (d-d') + d'$

$$= d = \dim k[x_1,\dots,x_d]$$

$$= \dim A \quad \text{as } A/k[x_1,\dots,x_d] \text{ is integral.}$$

$\blacksquare$