MATH200C, HOMEWORK 3

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Zariski topology

Suppose A is a unital commutative ring and $X := \mathrm{Spec}(A)$. For $f \in A$, let $X_f := X \setminus V(\langle f \rangle)$.

(1) Prove that $X_f = X_{f'}$ if and only if $\sqrt{\langle f \rangle} = \sqrt{\langle f' \rangle}$.
(2) Prove that there is a bijection between X_f and $\mathrm{Spec}(S_f^{-1}A)$ where $S_f := \{1, f, f^2, \ldots\}$. (We consider the spec of the zero ring to be empty.)
(3) Prove that $\{X_f | f \in A\}$ is a basis of open subsets of X.
(4) Prove that every open covering of X has a finite cover; that means if $X = \bigcup_{i\in I} \mathscr{O}_i$ where $\mathscr{O}_i$'s are open in X, then there is a finite subset J of I such that $X = \bigcup_{j\in J} \mathscr{O}_j$.

McCoy's result on finite union of ideals

Suppose $\mathfrak{a}, \mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_k \trianglelefteq A$,

$$\mathfrak{a} \subseteq \bigcup_{i=1}^{k} \mathfrak{b}_i, \text{ and } \mathfrak{a} \not\subseteq \bigcup_{1\le i\le k, i\neq j} \mathfrak{b}_i$$

for any $1 \le j \le k$. Then there is a positive integer n such that $\mathfrak{a}^n \subseteq \bigcap_{i=1}^{k} \mathfrak{b}_i$.

(**Hint**. Use strong induction on k; show that $\mathfrak{a} \subseteq \mathfrak{b}_1 \cup \mathfrak{b}_2$ implies $\mathfrak{a} \subseteq \mathfrak{b}_i$ for some i. Argue why $\mathfrak{b}_i \cap \mathfrak{a}$'s also satisfy the above conditions; and so W.L.O.G. we can assume that $\mathfrak{a} = \bigcup_{i=1}^{k} \mathfrak{b}_i$. By the strong induction hypothesis,

$$\mathfrak{a} \subseteq (\mathfrak{b}_1 + \mathfrak{b}_2) \cup \mathfrak{b}_3 \cup \cdots \cup \mathfrak{b}_k,$$

and similar inclusions for any pair $i \neq j$, deduce that $\mathfrak{a}^m \subseteq \prod_{1\le i<j\le k} (\mathfrak{b}_i + \mathfrak{b}_j)$ (I). Since $\mathfrak{a} = \bigcup_{i=1}^{k} \mathfrak{b}_i$, deduce that $\bigcap_{i\neq j} \mathfrak{b}_i = \bigcap_i \mathfrak{b}_i$ for any j (II). Show that claim follows from (I) and (II).)

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Ideal quotient.

Suppose $\mathfrak{a}, \mathfrak{b} \trianglelefteq A$; then $(\mathfrak{a} : \mathfrak{b}) := \{x \in A |\, x\mathfrak{b} \subseteq \mathfrak{a}\}$. Convince yourself that $(\mathfrak{a} : \mathfrak{b})$ is an ideal of A. Prove the following properties:

(1) $\mathfrak{a} \subseteq (\mathfrak{a} : \mathfrak{b})$.
(2) $(\mathfrak{a} : \mathfrak{b})\mathfrak{b} \subseteq \mathfrak{a}$.
(3) $((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}) = (\mathfrak{a} : \mathfrak{b}\mathfrak{c}) = ((\mathfrak{a} : \mathfrak{c}) : \mathfrak{b})$.
(4) $(\bigcap_i \mathfrak{a}_i : \mathfrak{b}) = \bigcap_i (\mathfrak{a}_i : \mathfrak{b})$.
(5) $(\mathfrak{a} : \sum_j \mathfrak{b}_j) = \bigcap_j (\mathfrak{a} : \mathfrak{b}_j)$.

Boolean ring as an example

Here you show that, a ring is locally reduced if and only if it is reduced (reduced means it has no non-zero nilpotent elements). Next you show that there are rings that are not integral domain; but they are locally integral domain. You also show being Noetherian is not a local property either.

(1) Suppose, for any $\mathfrak{p} \in \mathrm{Spec}(A)$, $\mathrm{Nil}(A_\mathfrak{p}) = 0$. Prove that $\mathrm{Nil}(A) = 0$. (**Hint**. Suppose $x \in \mathrm{Nil}(A)$ and consider $\mathrm{ann}(x) := \{a \in A |\, ax = 0\}$.)
(2) Suppose for any $x \in A$, $x^{n(x)} = x$ for some positive integer $n(x)$. Prove that $\mathrm{Spec}(A) = \mathrm{Max}(A)$.
(3) A ring A is called a Boolean ring if for any $a \in A$, $a^2 = a$.
   (a) Prove that, for any $\mathfrak{p} \in \mathrm{Spec}(A)$, $A/\mathfrak{p} \simeq \mathbb{Z}/2\mathbb{Z}$.
   (b) Prove that, for any $\mathfrak{p} \in \mathrm{Spec}(a)$, $A_\mathfrak{p} \simeq \mathbb{Z}/2\mathbb{Z}$. (**Hint**. If $a \in \mathfrak{p}$, then $1 - a \notin \mathfrak{p}$ and $\frac{a}{1} = \frac{a(1-a)}{1-a} = 0$.)
   (c) Let $A := P(X)$ be the power set of a non-empty set X. For $a_1, a_2 \in A$, let $a_1 + a_2 := a_1 \triangle a_2$ be the symmetric difference of a_1 and a_2; that means $a_1 \triangle a_2 := (a_1 \cup a_2) \setminus (a_1 \cap a_2)$. And $a_1 \cdot a_2 := a_1 \cap a_2$. Convince yourself that $(A, +, .)$ is a Boolean ring. Prove that A is Noetherian if and only if X is finite.
(4) Suppose, for any $\mathfrak{p} \in \mathrm{Spec}\, A$, $A_\mathfrak{p}$ is an integral domain. Is it true that A is an integral domain?
(5) Suppose, for any $\mathfrak{p} \in \mathrm{Spec}\, A$, $A_\mathfrak{p}$ is Noetherian. Is it true that A is Noetherian?