MATH200C, LECTURE 11

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Integral extension, going to a factor ring, and localization

Lemma 1. *Suppose B/A is an integral extension.*

1. *For $\mathfrak{b} \trianglelefteq B$, $A/\mathfrak{b}^c \hookrightarrow B/\mathfrak{b}$ is integral, where $\mathfrak{b}^c = \mathfrak{b} \cap A$.*
2. *For a multiplicatively closed subset S of A, $S^{-1}A \hookrightarrow S^{-1}A$ is integral.*

Proof. (1) For any $b \in B$, there is a monic polynomial $f(x) \in A[x]$ such that $f(b) = 0$. Then $\pi(f)(\pi(b)) = 0$, where π is induced by the natural quotient map $B \to B/\mathfrak{b}$.

(2) For any $b \in B$, there is a monic polynomial $f(x) \in A[x]$ such that $f(b) = 0$. Suppose $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$; then we have

$$b^n + a_{n-1}b^{n-1} + \cdots + a_0 = 0 \Rightarrow \left(\frac{b}{s}\right)^n + \frac{a_{n-1}}{s}\left(\frac{b}{s}\right)^{n-1} + \cdots + \frac{a_0}{s^n} = 0,$$

which implies that $\frac{b}{s}$ is integral over $S^{-1}A$ (coefficient of $(\frac{b}{s})^i$ is $\frac{a_i}{s^{n-i}}$). $\square$

Proposition 2. *Suppose B/A is a ring extension and C is the integral closure of A in B. Then $S^{-1}C$ is the integral closure of $S^{-1}A$ in $S^{-1}B$, where S is a multiplicatively closed subset of A..*

Proof. Suppose $\overline{C}$ is the integral closure of $S^{-1}A$ in $S^{-1}B$. Then by the previous lemma, we have that $S^{-1}C$ is a subring of $\overline{C}$. Suppose $\frac{b}{s}$ is integral over $S^{-1}A$; that means there are $a_i \in A$ and $s_i \in S$ such that

$$\left(\frac{b}{s}\right)^n + \frac{a_{n-1}}{s_{n-1}}\left(\frac{b}{s}\right)^{n-1} + \cdots + \frac{a_0}{s_0} = 0.$$

This implies that for some $s'' \in S$ such that

$$s''(s'b^n + \underbrace{s'_{n-1}sa_{n-1}}_{a'_{n-1}}b^{n-1} + \cdots + \underbrace{s'_i s^{n-i}a_i}_{a'_i}b^i + \cdots + \underbrace{s'_0 s^n a_0}_{a'_0}) = 0$$

where $s'_i := s_0 \cdots s_{i-1}s_{i+1} \cdots s_{n-1}$ and $s' := \prod_{i=0}^{n-1} s_i$. Hence

$$(s''s'b)^n + a'_{n-1}(s''s'b)^{n-1} + \cdots + (s''s')^{n-i-1}a_i(s'b)^i + \cdots + (s''s')^{n-1}a'_0 = 0,$$

which implies that $s''s'b$ is integral over A. Therefore $s''s'b \in C$. Thus $\frac{b}{s} = \frac{s''s'b}{s''s's} \in S^{-1}C$; and claim follows. □

Being integrally closed is a local property

Theorem 3. *Suppose D is an integral domain. Then the following statements are equivalent:*

(1) D is integrally closed.
(2) For any $\mathfrak{p} \in \operatorname{Spec}(D)$, $D_{\mathfrak{p}}$ is integrally closed.
(3) For any $\mathfrak{m} \in \operatorname{Max}(D)$, $D_{\mathfrak{m}}$ is integrally closed.

Proof. (1)⇒(2) Suppose K is the field of fractions of D. Then the integral closure of D in K is D. Hence by the previous proposition, $D_{\mathfrak{p}}$ is the integral closure of $D_{\mathfrak{p}}$ in $S_{\mathfrak{p}}^{-1}K = K$; and claim follows.

(2)⇒(3) is clear.

(3)⇒(1) Suppose C is the integral closure of D in K, where K is the field of fractions of D. Then by the previous proposition, for any $\mathfrak{m} \in \operatorname{Max}(D)$, $S_{\mathfrak{m}}^{-1}C$ is the integral closure of $D_{\mathfrak{m}}$ in K where $S_{\mathfrak{m}} := D \setminus \mathfrak{m}$. By our assumption, we have that $S_{\mathfrak{m}}^{-1}C = S_{\mathfrak{m}}^{-1}D$ for any $\mathfrak{m} \in \operatorname{Max}(D)$. Think about C as a D-module; so the injection $i : D \to C, i(x) := x$ is a D-module homomorphism and for any $\mathfrak{m} \in \operatorname{Max}(D)$, $i_{\mathfrak{m}}$ is surjective. We have seen that this implies i is surjective (this implies that $(C/D)_{\mathfrak{m}} = 0$ where C/D is considered as a D-module; from here we deduce that $\operatorname{Ann}(C/D) = D$). □

The contraction map of an integral embedding

Lemma 4. *Suppose B/A is an integral extension, and B is an integral domain. Then A is a field if and only if B is a field.*

Proof. (⇒) For any $b \in B$, $A[b]$ is a finitely generated A-module; and so $A[b]$ is a finite dimensional A-algebra and it is an integral domain. This implies that $A[b]$ is a field. For $x \in A[b]$, let $l_x : A[b] \to A[b], l_x(y) := xy$. Then l_x is an A-linear map. Since $A[b]$ is an integral domain, l_x is injective when x is not zero. An injective linear map on a finite dimensional vector space is invertible. Hence l_x is surjective, which implies that x is a unit; and claim follows. Hence $b^{-1} \in A[b] \subseteq B$, and claim follows.

($\Leftarrow$) For $a \in A$, there is $a^{-1} \in B$. Since B/A is integral, there are $a_i \in A$ such that

$$a^{-n} + a_{n-1}a^{-(n-1)} + \cdots + a_0 = 0.$$

Hence

$$a^{-1} = -(a_{n-1} + a_{n-2}a + \cdots + a_0 a^{n-1}) \in A;$$

and claim follows. □

Corollary 5. *Suppose $f : A \hookrightarrow B$ is integral; then for any $\mathfrak{q} \in \operatorname{Spec}(B)$,*

$$f^*(\mathfrak{q}) \in \operatorname{Max}(A) \Leftrightarrow \mathfrak{q} \in \operatorname{Max}(B).$$

Proof. For any $\mathfrak{q} \in \operatorname{Spec}(B)$, $A/f^*(\mathfrak{q}) \hookrightarrow B/\mathfrak{q}$ is an integral extension and $B/\mathfrak{q}$ is an integral domain. Hence by the previous lemma, $A/f^*(\mathfrak{q})$ is a field if and only if $B/\mathfrak{q}$ is a field; and claim follows as we have

$$f^*(\mathfrak{q}) \in \operatorname{Max}(A) \Leftrightarrow A/f^*(\mathfrak{q}) \text{ is a field} \Leftrightarrow B/\mathfrak{q} \text{ is a field} \Leftrightarrow \mathfrak{q} \in \operatorname{Max}(B).$$

□

Proposition 6. *Suppose $f : A \hookrightarrow B$ is integral; then $f^* : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is onto, and $f^*(\operatorname{Max}(B)) = \operatorname{Max}(A)$.*

Proof. Suppose $\mathfrak{p} \in A$; and let $S_\mathfrak{p} := A \setminus \mathfrak{p}$. Then $f_\mathfrak{p} : S_\mathfrak{p}^{-1}A \hookrightarrow S_\mathfrak{p}^{-1}B$ is integral. Hence by the previous corollary $f_\mathfrak{p}^*(\operatorname{Max}(S_\mathfrak{p}^{-1}B)) \subseteq \operatorname{Max}(S_\mathfrak{p}^{-1}A) = \{S_\mathfrak{p}^{-1}\mathfrak{p}\}$. This implies that there is a prime ideal $\mathfrak{q}$ of B such that

1. (1) $\mathfrak{q} \cap S_\mathfrak{p} = \varnothing$; (this implies $\mathfrak{q} \cap A \subseteq \mathfrak{p}$.)
2. (2) $S_\mathfrak{p}^{-1}\mathfrak{q} \cap S_\mathfrak{p}^{-1}A = S_\mathfrak{p}^{-1}\mathfrak{p}$. (this implies for any $x \in \mathfrak{p}$, $\frac{x}{1} \in S_\mathfrak{p}^{-1}\mathfrak{q}$; and so $x \in \mathfrak{q}$ as $S_\mathfrak{p}(\mathfrak{q}) = \mathfrak{q}$.)

Hence $f^*(\mathfrak{q}) = \mathfrak{q} \cap A = \mathfrak{p}$, which implies that f^* is surjective.

To show the second part, we notice that we have already proved $f^*(\operatorname{Max}(B)) \subseteq \operatorname{Max}(A)$. For $\mathfrak{m} \in \operatorname{Max}(A)$, there is $\mathfrak{q} \in \operatorname{Spec}(B)$ such that $f^*(\mathfrak{q}) = \mathfrak{m}$ as f^* is onto. By the previous corollary, since $f^*(\mathfrak{q})$ is maximal, we have that $\mathfrak{q}$ is maximal; and claim follows. □

Corollary 7. *Suppose $f : A \hookrightarrow B$ is integral; then $f^* : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is a closed map; in fact $f^*(V(\mathfrak{b})) = V(\mathfrak{b}^c)$ for any $\mathfrak{b} \trianglelefteq B$.*

Proof. For $\mathfrak{b}\trianglelefteq B$, let $\overline{f}: A/\mathfrak{b}^c \to B/\mathfrak{b}$, $\overline{f}(a+\mathfrak{b}^c) := f(a)+\mathfrak{b}$. Then $\overline{f}$ is integral; and so $\overline{f}^*$ is surjective. Let $\pi_{\mathfrak{b}}: B \to B/\mathfrak{b}$ and $\pi_{\mathfrak{b}^c}: A \to A/\mathfrak{b}^c$ be the natural quotient maps; then we have the following commuting diagram consisting of bijections and onto maps:

$$\begin{array}{ccc} \mathrm{Spec}(B/\mathfrak{b}) & \xrightarrow{\pi_{\mathfrak{b}}^*} & V(\mathfrak{b}) \\ \downarrow{\scriptstyle \overline{f}^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{Spec}(A/\mathfrak{b}^c) & \xrightarrow{\pi_{\mathfrak{b}^c}^*} & V(\mathfrak{b}^c) \end{array}$$

. This implies that $f^*(V(\mathfrak{b})) = V(\mathfrak{b}^c)$; and claim follows. ☐

Going-Up theorem

Theorem 8. *Suppose* $f: A \hookrightarrow B$ *is integral,* $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ *is a chain in* $\mathrm{Spec}(A)$*, and* $\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_m$ *is a chain in* $\mathrm{Spec}(B)$ *such that* $f^*(\mathfrak{q}_i) = \mathfrak{p}^i$*. Then there are* $\underbrace{\mathfrak{q}_{m+1} \subsetneq \cdots \subsetneq \mathfrak{q}_n}_{\overrightarrow{\textit{Going-Up}}}$ *in* $\mathrm{Spec}(B)$ *such that* $f^*(\mathfrak{q}_i) = \mathfrak{p}_i$ *and* $\mathfrak{q}_m \subsetneq \mathfrak{q}_{m+1}$.

Proof. Inductively on m, we show the existence of $\mathfrak{q}_{m+1}$. The base case of $m = -1$ is a consequence of surjectivity of f^*. So we focus on the induction step. By the previous corollary, $f^*(V(\mathfrak{q}_m)) = V(\mathfrak{p}_m)$; and so there is $\mathfrak{q}_{m+1} \in V(\mathfrak{q}_m)$ such that $f^*(\mathfrak{q}_{m+1}) = \mathfrak{p}_{m+1}$ as $\mathfrak{p}_{m+1} \in V(\mathfrak{p}_m)$. Since $f^*(\mathfrak{q}_{m+1}) \neq f^*(\mathfrak{q}_m)$, we have that $\mathfrak{q}_m \neq \mathfrak{q}_{m+1}$; and claim follows. ☐

Next we show $\dim(f^*)^{-1}(\mathfrak{p}) = 0$ for any $\mathfrak{p} \in \mathrm{Spec}(A)$ if $f: A \hookrightarrow B$ is integral:

Lemma 9. *Suppose* $f: A \hookrightarrow B$ *is integral,* $\mathfrak{p} \in \mathrm{Spec}(A)$*,* $\mathfrak{q}_1 \subseteq \mathfrak{q}_2 \in \mathrm{Spec}(B)$*, and* $f^*(\mathfrak{q}_1) = f^*(\mathfrak{q}_2) = \mathfrak{p}$*. Then* $\mathfrak{q}_1 = \mathfrak{q}_2$.

We will be using the above lemma and Going-Up Theorem to show $\dim A = \dim B$ is B/A is integral.

(We will continue in the next lecture.)