MATH200C, LECTURE 13

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Integral morphisms are open under some conditions.

Suppose $f : A \hookrightarrow B$ is an integral embedding; so far we have proved that f^* is onto and closed, any fiber has dimension 0, and $\dim A = \dim B$. Next we proved the Going-Down Theorem under some conditions. Today we show that under the same conditions f^* is also open.

Theorem 1. *Suppose $f : A \hookrightarrow B$ is integral, B is an integral domain, and A is integrally closed. Then $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is open.*

Proof. We know that $\{\mathscr{O}_b\}_{b\in B}$ forms a basis for the open subsets of $\mathrm{Spec}(B)$ where $\mathscr{O}_b := \{\mathfrak{q} \in \mathrm{Spec}(B) |\, b \notin \mathfrak{q}\}$. So it is enough to show $f^*(\mathscr{O}_b)$ is open for any $b \in B$. Take $b \in B$, and let $g(x)$ be the minimal polynomial of b over the field of fractions of A; say $g(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$. In the previous lecture, we have proved that $a_i \in A$.

Claim. $f^*(\mathscr{O}_b) = \bigcup_{i=0}^{n-1} \mathscr{O}_{a_i}$; and so $f^*(\mathscr{O}_b)$ is open.

Proof of Claim. ($\subseteq$) Suppose $\mathfrak{p} \in f^*(\mathscr{O}_b)$. Then there is $\mathfrak{q} \in \mathrm{Spec}(B)$ such that $\mathfrak{q}^c = \mathfrak{p}$; and so $\mathfrak{p}^e \subseteq \mathfrak{q}$, which implies that $\sqrt{\mathfrak{p}^e} \subseteq \mathfrak{q}$. Therefore knowing that $b \notin \mathfrak{q}$ implies that $b \notin \sqrt{\mathfrak{p}^e}$. By an earlier result, we get that b is not integral over $\mathfrak{p}$; and so at least one of the a_i's is not in $\mathfrak{p}$, which means $\mathfrak{p} \in \bigcup_{i=0}^{n-1} \mathscr{O}_{a_i}$.

($\supseteq$) Suppose $\mathfrak{p} \in \bigcup_{i=0}^{n-1}$; by a lemma that we proved in the previous lecture, if b is integral over $\mathfrak{p}$, then all the non-leading coefficients of the minimal polynomial of b over the field of fractions of A should be in $\mathfrak{p}$. So we deduce that b is not integral over $\mathfrak{p}$. Thus by a proposition that was proved in the previous lecture, $b \notin \sqrt{\mathfrak{p}^e}$. Hence there is $\widetilde{\mathfrak{q}} \in \mathrm{Spec}(B)$ such that $\mathfrak{p}^e \subseteq \widetilde{\mathfrak{q}}$ and $b \notin \widetilde{\mathfrak{q}}$. So we have $\mathfrak{p} \subseteq \widetilde{\mathfrak{q}}^c$ is a chain in $\mathrm{Spec}(A)$; therefore by the Going-Down Theorem, there is $\mathfrak{q} \in \mathrm{Spec}(B)$ such that $\mathfrak{q} \subseteq \widetilde{\mathfrak{q}}$ and $\mathfrak{q}^c = \mathfrak{p}$. Hence $f^*(\mathfrak{q}) = \mathfrak{p}$ and $b \notin \mathfrak{q}$ as $\mathfrak{q} \subseteq \widetilde{\mathfrak{q}}$ and $b \notin \widetilde{\mathfrak{q}}$; this means $\mathfrak{p} \in f^*(\mathscr{O}_b)$. □

Getting Noetherian condition for some integral closures.

As it has been pointed out earlier, one of the important examples that you should have in mind is the integral closure $\mathscr{O}_k$ of $\mathbb{Z}$ in a number field k. So far we have proved that $f^*: \mathrm{Spec}(\mathscr{O}_k) \to \mathrm{Spec}(\mathbb{Z})$ is an open, closed, and onto map. And $\dim \mathscr{O}_k = \dim \mathbb{Z} = 1$. Next we want to show they are Noetherian. We do it in much more generality.

Theorem 2. *Suppose $f: A \hookrightarrow B$ is integral, B is an integral domain, and A is integrally closed. Let F be the field of fractions of A, and E be the field of fractions of B. Suppose E/F is a finite separable extension. Then there are $e_1, \ldots, e_n \in E$ such that*

$$B \subseteq Ae_1 + \cdots + Ae_n. \tag{1}$$

In particular, if A is Noetherian, then B is Noetherian.

Before we prove the claimed inclusion (1) in above theorem, we show how this implies the claimed Noetherian condition.

Proof of the Noetherian condition. If A is Noetherian, then any finitely generated A-module is a Noetherian A-module. Hence $\sum_{i=1}^n Ae_i$ is a Noetherian A-module. This implies that any of its A-submodules is Noetherian; and so B is a Noetherian A-module. Therefore B is a Noetherian B-module, which means B is Noetherian. □

To show the above theorem, first we review some basic properties of finite separable field extensions and non-degenerate bilinear forms.

Recall from linear algebra. Suppose V is a finite dimensional vector space over a field F. Let $\mathfrak{B} := \{v_1, \ldots, v_n\}$ be an F-basis of V. For $v \in V$, we let $|v\rangle_{\mathfrak{B}} := \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ if $v = \sum_{i=1}^n c_i v_i$; and we let $\langle v|_{\mathfrak{B}}$ be the transpose of $|v\rangle_{\mathfrak{B}}$. For any F-linear map $T: V \to V$, we have a matrix $[T]_{\mathfrak{B}} \in M_n(F)$ such that for any $v \in V$, $|T(v)\rangle_{\mathfrak{B}} = [T]_{\mathfrak{B}} |v\rangle_{\mathfrak{B}}$.

Lemma 3. *Suppose E/F is finite separable field extension. Let $\mathfrak{B} := \{e_1, \ldots, e_n\}$ be an F-basis of E, and $l_e: E \to E, l_e(e') := ee'$. Let $\{\sigma_1, \ldots, \sigma_n\}$ be the set of*

F-embeddings of E into an algebraic closure $\overline{F}$ of F. Then for any $e \in E$, $[l_e]_{\mathfrak{B}}$ is similar to the diagonal matrix $\operatorname{diag}(\sigma_1(e), \ldots, \sigma_n(e))$ *over $\overline{F}$.*

Proof. Since E/F is a finite separable extension, $E = F[\alpha]$ for some $\alpha \in E$. Let $m_{\alpha,F}(x)$ be the minimal polynomial of α over F. So $[E : F] = \deg m_{\alpha,F}$. Since E/F is a separable extension, $m_{\alpha,F}(x)$ has $n := [E : F]$ distinct zeros; say $m_{\alpha,F}(x) = \prod_{i=1}^{n}(x - \alpha_i)$ for α_i. Then for any $\sigma \in \operatorname{Embed}_F(E, \overline{F})$, $\sigma(\alpha) \in \{\alpha_1, \ldots, \alpha_n\}$; and for any i, there is a unique F-embedding σ_i of E into $\overline{F}$ that sends α to α_i. Hence after rearranging we can and will assume that $\sigma_i(\alpha) = \alpha_i$. We have

$$\begin{aligned} E \otimes_F \overline{F} =& F[\alpha] \otimes_F \overline{F} \\ \simeq& F[x]/\langle \prod_{i=1}^{n}(x - \alpha_i)\rangle \otimes_F \overline{F} \simeq \overline{F}[x]/\langle \prod_{i=1}^{n}(x - \alpha_i)\rangle \\ \simeq& \bigoplus_{i=1}^{n} \overline{F}[x]/\langle x - \alpha_i\rangle \simeq \bigoplus_{i=1}^{n} \overline{F}. \end{aligned}$$

And following these isomorphisms we have that

$$\begin{aligned} \alpha \otimes 1 \mapsto& x + \mathfrak{a} \otimes 1 \mapsto x + \mathfrak{a}^e \\ \mapsto& (x + \langle x - \alpha_1\rangle, \ldots, x + \langle x - \alpha_n\rangle) \mapsto (\alpha_1, \ldots, \alpha_n) \\ =& (\sigma_1(\alpha), \ldots, \sigma_n(\alpha)). \end{aligned}$$

Since the above isomorphism is an $\overline{F}$-algebra isomorphism and $E = F[\alpha]$, we get an $\overline{F}$-algebra isomorphism

$$\theta : E \otimes_F \overline{F} \to \bigoplus_{i=1}^{n} \overline{F}, \theta(e \otimes 1) = (\sigma_1(e), \ldots, \sigma_n(e)),$$

for any $e \in E$. Therefore we get the following commuting diagram

$$\begin{array}{ccccc} E & \hookrightarrow & E \otimes_F \overline{F} & \xrightarrow{\theta} & \overline{F}^n \\ \downarrow{\scriptstyle l_e} & & \downarrow{\scriptstyle l_e \otimes \mathrm{id}_{\overline{F}}} & & \downarrow{\scriptstyle d_e} \\ E & \hookrightarrow & E \otimes_F \overline{F} & \xrightarrow{\theta} & \overline{F}^n \end{array}$$

where $d_e : \overline{F}^n \to \overline{F}^n, d_e(x_1, \ldots, x_n) := (\sigma_1(e)x_1, \ldots, \sigma_n(e)x_n)$. Notice that since $\{e_1, \ldots, e_n\}$ is an F-basis of E, $\{e_1 \otimes 1, \ldots, e_n \otimes 1\}$ is an $\overline{F}$-basis of $E \otimes_F \overline{F}$ and $\widehat{\mathfrak{B}} := \{\theta(e_1), \ldots, \theta(e_n)\}$ is an $\overline{F}$-basis of $\overline{F}^n$. As θ is an $\overline{F}$-algebra isomorphism, by the above diagram $[l_e]_{\mathfrak{B}} = [d_e]_{\widehat{\mathfrak{B}}}$. On the other hand, in the standard basis $\mathfrak{B}'$ of $\overline{F}^n$ we have $[d_e]_{\mathfrak{B}'} = \mathrm{diag}(\sigma_1(e), \ldots, \sigma_n(e))$; and claim follows. □

Corollary 4. *Suppose E/F is a finite separable extension. Let*

$$\mathrm{Tr}_{E/F}(e) := \sum_{\sigma \in \mathrm{Embed}_F(E, \overline{F})} \sigma(e),$$

and

$$\mathrm{N}_{E/F}(e) := \prod_{\sigma \in \mathrm{Embed}_F(E, \overline{F})} \sigma(e).$$

Let $l_e : E \to E, l_e(e') := ee'$. Then $\mathrm{Tr}_{E/F}(e) = \mathrm{Tr}(l_e)$ and $\mathrm{N}_{E/F}(e) = \det l_e$; in particular, $\mathrm{Tr}_{E/F}(E) \subseteq F$ and $\mathrm{N}_{E/F}(E) \subseteq F$.

Note. Suppose $\mathfrak{B} := \{e_1, \ldots, e_n\}$ is an F-basis of a vector space V; and $f : V \times V \to F$ is a bilinear map. Then $[f]_{\mathfrak{B}} := [f(e_i, e_j)]$, and for any $v, w \in V$, we have $f(v, w) = \langle v|_{\mathfrak{B}} [f]_{\mathfrak{B}} |w\rangle_{\mathfrak{B}}$.

Lemma 5. *In the above setting, f is non-degenerate if and only if $\det[f]_{\mathfrak{B}} \neq 0$.*

Proof. ($\Rightarrow$) suppose $\det[f]_{\mathfrak{B}} = 0$; then there is $w \neq 0$ such that $[f]_{\mathfrak{B}}|w\rangle_{\mathfrak{B}} = 0$; and so for any $v \in V$, $f(v, w) = 0$ and $w \neq 0$, which contradicts the assumption that f is non-degenerate.

($\Leftarrow$) suppose f is degenerate; so there is $w \neq 0$ such that $f(V, w) = 0$; this implies that $\langle v|_{\mathfrak{B}} [f]_{\mathfrak{B}} |w\rangle_{\mathfrak{B}} = 0$ for any $v \in V$. Letting $v = e_i$, we deduce that the i-th component of $[f]_{\mathfrak{B}}|w\rangle_{\mathfrak{B}}$ is zero. Therefore $[f]_{\mathfrak{B}}|w\rangle_{\mathfrak{B}} = 0$. As $\det[f]_{\mathfrak{B}} \neq 0$, we deduce that $w = 0$, which is a contradiction. □

Lemma 6. *Suppose V is a finite dimensional F-vector space, and $f : V \times V \to F$ is a non-degenerate F-bilinear form; then $T_f : V \to V^*, (T_f(v))(w) := f(v, w)$ is an F-module isomorphism, where $V^* := \mathrm{Hom}_F(V, F)$.*

Proof. Since f is linear in the second factor, $T_f(v) \in V^*$; and since f is linear in the first factor, $v \mapsto T_f(v)$ is a linear map. If $v \in \ker T_f$, then for any $w \in V$, $(T_f(v))(w) = 0$, which implies that $f(v, V) = 0$; and so $v = 0$ as f is non-degenerate. Hence T_f is an injective F-linear map. On the other hand, $V^* = \mathrm{Hom}_F(\bigoplus_{i=1}^n F, F) \simeq \bigoplus_{i=1}^n \mathrm{Hom}_F(F, F) \simeq F^n \simeq V$. Hence T_f is also surjective as V and V^* have equal dimensions. □

Lemma 7. *Suppose* $\{v_1, \ldots, v_n\}$ *is an* F*-basis of* V*, and* $f : V \times V \to F$ *is a non-degenerate bilinear map. Then there is a dual basis* $\{w_1, \ldots, w_n\}$ *with respect to* f*; that means it is a basis and*

$$f(v_i, w_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\{v_1^*, \ldots, v_n^*\}$ be the dual basis of V^*; that means $v_i^* : V \to F$ is the F-linear extension of $v_i^*(v_j) := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$ By the previous lemma, T_f is surjective; and so there are w_i's in V such that $T_f(w_i) = v_i^*$; and claim follows. □

Lemma 8. *Suppose* E/F *is a finite separable field extension. Then* $f(e, e') := \mathrm{Tr}_{E/F}(ee')$ *is a non-degenerate symmetric bilinear form on* E*.*

Proof. Suppose $\{e_1, \ldots, e_n\}$ is an F-basis of E. Then we have to show

$$\det[\mathrm{Tr}_{E/F}(e_i e_j)] \neq 0.$$

We notice that $\mathrm{Tr}_{E/F}(e_i e_j) = \sum_{k=1}^n \sigma_k(e_i e_j) = \sum_{k=1}^n \sigma_k(e_i)\sigma(e_j)$ where $\mathrm{Embed}_F(E, \overline{F})$. Let $X := [\sigma_k(e_i)]$ (the ik-th entry is $\sigma_k(e_i)$). Then by the previous equality we have

$$[\mathrm{Tr}_{E/F}(e_i e_j)] = XX^t; \text{ and so } \det[\mathrm{Tr}_{E/F}(e_i e_j)] = \det X^2.$$

Hence it is enough to show rows of X are linearly independent. This we have already pointed out in the proof of Lemma 3: $\{\theta(e_1), \ldots, \theta(e_n)\}$ is an $\overline{F}$-basis of $\overline{F}^n$. □

We will prove Theorem 3 in the next lecture.