1 Homework 6.

1. Suppose $f : A \to B$ is a ring homomorphism. For $\mathfrak{p} \in \operatorname{Spec}(A)$, let $S_\mathfrak{p} := A \setminus \mathfrak{p}$, $A_\mathfrak{p} := S_\mathfrak{p}^{-1}A$, $k(\mathfrak{p}) := A_\mathfrak{p}/S_\mathfrak{p}^{-1}\mathfrak{p}$, and $B_\mathfrak{p} := f(S_\mathfrak{p})^{-1}B$. Prove that

$$B \otimes_A k(\mathfrak{p}) \simeq \frac{B_\mathfrak{p}}{f(S_\mathfrak{p})^{-1}\mathfrak{p}^e}.$$

(**Hint.** Show that $b \otimes (\frac{a}{s} + S_\mathfrak{p}^{-1}\mathfrak{p}) \mapsto \frac{f(a)b}{f(s)} + f(S_\mathfrak{p})^{-1}\mathfrak{p}^e$ is a well-defined ring homomorphism. Next, consider the function from $B_\mathfrak{p}$ to $B \otimes_A k(\mathfrak{p})$ given by $\frac{b}{f(s)} \mapsto b \otimes \frac{1}{s}$; notice that if $s_1, s_2 \in S$ and $f(s_1) = f(s_2)$, then

$$b \otimes \frac{1}{s_1} - b \otimes \frac{1}{s_2} = b \otimes \frac{s_2 - s_1}{s_1 s_2} = bf(s_2 - s_1) \otimes \frac{1}{s_1 s_2} = 0,$$

and so the given function is well-defined. Check that it is a ring homomorphism, and $f(S_\mathfrak{p})^{-1}\mathfrak{p}^e$ is in its kernel. Therefore,

$$\frac{b}{f(s)} + f(S_\mathfrak{p})^{-1}\mathfrak{p}^e \mapsto b \otimes \frac{1}{s}$$

is a well-defined ring homomorphism. Show that these given ring homomorphisms are inverse of each other.)

(**Remark**. In class, we showed that there exists a natural bijection between the fiber $(f^*)^{-1}(\mathfrak{p})$ and $\operatorname{Spec}(\frac{B_\mathfrak{p}}{f(S_\mathfrak{p})^{-1}\mathfrak{p}^e})$. Based on this exercise, you see that there exists a natural bijection between $(f^*)^{-1}(\mathfrak{p})$ and $\operatorname{Spec}(B \otimes_A k(\mathfrak{p}))$. This can be better seen as the fiber product of $\operatorname{Spec}(B)$ and $\operatorname{Spec}(k(\mathfrak{p}))$ over $\operatorname{Spec}(A)$!)

2. Suppose $\mathfrak{a}, \mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_k \trianglelefteq A$,

$$\mathfrak{a} \subseteq \bigcup_{i=1}^{k} \mathfrak{b}_i, \text{ and } \mathfrak{a} \not\subseteq \bigcup_{1 \le i \le k, i \ne j} \mathfrak{b}_i$$

for every $1 \le j \le k$. Then there exists a positive integer n such that $\mathfrak{a}^n \subseteq \bigcap_{i=1}^{k} \mathfrak{b}_i$.

(**Hint**. Use strong induction on k; show that $\mathfrak{a} \subseteq \mathfrak{b}_1 \cup \mathfrak{b}_2$ implies $\mathfrak{a} \subseteq \mathfrak{b}_i$ for some i. Show the claim for $k = 3$ as well. Argue why $\mathfrak{b}_i \cap \mathfrak{a}$'s also satisfy the above conditions; and so W.L.O.G. we can assume that $\mathfrak{a} = \bigcup_{i=1}^{k} \mathfrak{b}_i$. Let $\mathfrak{c} := \bigcap_{i=1}^{k} \mathfrak{b}_i$. Show that for every j

$$\mathfrak{c} = \bigcap_{i \ne j} \mathfrak{b}_i.$$

To show this, let $x_j \in \mathfrak{b}_j \setminus (\bigcup_{i\neq j} \mathfrak{b}_i)$ and $y \in \bigcap_{i\neq j} \mathfrak{b}_i$; notice that $y + x_j \in \mathfrak{a}$ and $y + x_j \notin \bigcup_{i\neq j} \mathfrak{b}_i$. Deduce that $y + x_j \in \mathfrak{b}_j$, and so $y \in \mathfrak{b}_j$.

Next use the strong induction hypothesis, the inclusion

$$\mathfrak{a} \subseteq (\mathfrak{b}_1 + \mathfrak{b}_2) \cup \mathfrak{b}_3 \cup \cdots \cup \mathfrak{b}_k,$$

and similar inclusions for every pair $i \neq j$, and deduce that

$$\mathfrak{a}^m \subseteq \prod_{1\leq i<j\leq k} (\mathfrak{b}_i + \mathfrak{b}_j) \quad \text{(I)},$$

for some positive integer m. Notice that if the right hand side of (I) is expanded, every term is in a product of at least $n-1$ $\mathfrak{b}_i$'s, and so the right hand side is a subset of $\mathfrak{c}$.)

(**Remark**. This result is due to McCoy.)

3. Suppose A and B are two unital commutative rings, and $f : A \to B$ is a ring homomorphism. Consider B as an A-module where $a.b := f(a)b$. Suppose B is a flat A-module. Prove that the following statements are equivalent:

   (a) For every $\mathfrak{a} \trianglelefteq A$, $\mathfrak{a}^{ec} = \mathfrak{a}$.

   (b) $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.

   (c) For every $\mathfrak{m} \in \mathrm{Max}(A)$, $\mathfrak{m}^e \neq B$.

   (d) If M is every non-zero A-module, then $M \otimes_A B \neq 0$.

   (e) For every A-module M, $\theta : M \to M \otimes_A B, \theta(x) := x \otimes 1$ is injective.

   (**Hint.** (1)⇒(2), in class we proved that $\mathfrak{p}$ is in the image of f^* if and only if $\mathfrak{p}^{ec} = \mathfrak{p}$.

   (2)⇒(3), if $f^*(\mathfrak{p}') = \mathfrak{m}$, then $\mathfrak{m}^e \subseteq \mathfrak{p}'$.

   (3)⇒(4), For any $x \in M$, $0 \to Ax \to M$ is exact. Since B is flat,

$$0 \to Ax \otimes_A B \to M \otimes_A B$$

   is exact. So to show $M \otimes_A B$ is not zero, it is enough to show $Ax \otimes_A B$ is not zero. Suppose $\mathfrak{a} := \{a \in A |\, ax = 0\}$; then $Ax \simeq A/\mathfrak{a}$ as an A-module. Hence $Ax \otimes_A B \simeq B/\mathfrak{a}^e$ as an A-module. Suppose $\mathfrak{m}$ is a maximal ideal such that $\mathfrak{a} \subseteq \mathfrak{m}$, and deduce the claim.

(4)⇒(5), Suppose $M' := \ker\theta$. Since B is a flat A-module,

$$0 \to M' \otimes_A B \to M \otimes_A B \xrightarrow{g} (M \otimes_A B) \otimes_A B$$

is exact, where $g := \theta \otimes \mathrm{id}_B$. View $M \otimes_A B$ as an B-module and let

$$h : (M \otimes_A B) \otimes_A B \to M \otimes_A B, h(x \otimes b) := xb.$$

Show that h is a well-defined B-module homomorphism. Notice that

$$g(m \otimes b) = \theta(m) \otimes b = (m \otimes 1) \otimes b;$$

and so $(h \circ g)(m \otimes b) = (m \otimes 1)b = m \otimes b$. This implies that $h \circ g = \mathrm{id}$. Deduce that g is injective.

(5)⇒(1) Show that $\overline{f} : A/\mathfrak{a} \to B/\mathfrak{a}^e, \overline{f}(a + \mathfrak{a}) := f(a) + \mathfrak{a}^e$ is a well-defined injective ring homomorphism.)

Remark. We say the extension B/A is *faithfully flat* if the above statements hold.

4. Suppose $f : A \to B$ is a ring homomorphism and B is a flat A-module. Suppose $\mathfrak{q} \in \mathrm{Spec}(B)$ and $\mathfrak{p} := \mathfrak{q}^c \in \mathrm{Spec}(A)$. Let $S_\mathfrak{p} := A \backslash \mathfrak{p}$ and $S_\mathfrak{q} := B \backslash \mathfrak{q}$. Let $A_\mathfrak{p} := S_\mathfrak{p}^{-1}$, $B_\mathfrak{p} := S_\mathfrak{p}^{-1} B$, and $B_\mathfrak{q} := S_\mathfrak{q}^{-1} B$. Recall that $B_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module. Notice that $B_\mathfrak{p}$ can be viewed as a subring of $B_\mathfrak{q}$, and it is a localization of $B_\mathfrak{p}$, and so $B_\mathfrak{q}$ is a flat $B_\mathfrak{p}$-module. Hence, we can view $B_\mathfrak{q}$ as a flat $A_\mathfrak{p}$-module. Let $g : A_\mathfrak{p} \to B_\mathfrak{q}$ be the corresponding ring homomorphism. Prove that

$$g^* : \mathrm{Spec}(B_\mathfrak{q}) \to \mathrm{Spec}(A_\mathfrak{p})$$

is surjective.

(**Hint.** Use part (c) of the previous problem, and show that $g : A_\mathfrak{p} \to B_\mathfrak{q}$ is faithfully flat.)

5. Suppose A is a unital commutative ring and G is a finite subgroup of the group automorphisms of A. Let

$$A^G := \{a \in A \mid \forall g \in G, g(a) = a\}.$$

(a) Prove that A/A^G is an integral extension.

(b) Suppose $\mathfrak{p} \in \operatorname{Spec}(A^G)$. Prove that G acts transitively on

$$\{\mathfrak{q} \in \operatorname{Spec}(A) \mid \mathfrak{q}^c = \mathfrak{p}\};$$

in particular, this is a finite set. (This means $\operatorname{Spec}(A) \to \operatorname{Spec}(A^G)$ has finite fibers.)

(**Hint.** Notice that $a \in A$ is a zero of

$$f_{a;G}(x) := \prod_{g \in G} (x - g(a)).$$

Argue why $f_{a;G}(x) \in A^G[x]$.

Suppose $\mathfrak{q}_1^c = \mathfrak{q}_2^c = \mathfrak{p}$. For every $a \in \mathfrak{q}_1$, we have

$$N_G(a) := \prod_{g \in G} g(a) \in A^G \cap \mathfrak{q}_1.$$

Hence, for all $a \in \mathfrak{q}_1$, $N_G(a) \in \mathfrak{q}_2$. Obtain that for all $a \in \mathfrak{a}_1$, $g(a) \in \mathfrak{q}_2$ for some $g \in G$. Hence,

$$\mathfrak{q}_1 \subseteq \bigcup_{g \in G} g(\mathfrak{q}_2).$$

Deduce that $\mathfrak{q}_1 = g(\mathfrak{q}_2)$ for some $g \in G$.)

6. Suppose A is an integrally closed integral domain, F is a field of fractions of A, and E is a finite Galois extension of F. Let $G := \operatorname{Gal}(E/F)$. Let B be the integral closure of A in E.

   (a) Prove that for every $\sigma \in G$, $\sigma(B) = B$.

   (b) Prove that $A = B^G$.

7. Suppose F is a field. Let $\sigma_j(x_1, \ldots, x_n) \in A := F[x_1, \ldots, x_n]$ be such that

$$(T - x_1) \cdots (T - x_n) = \sum_{j=0}^{n} (-1)^j \sigma_j(x_1, \ldots, x_n) T^{n-j} \in A[T].$$

Notice that the symmetric group S_n is a subgroup of the group of automorphisms of A. Let $L := F(x_1, \ldots, x_n)$ and $K := F(\sigma_1, \ldots, \sigma_n)$. Recall that $K = \operatorname{Fix}(S_n)$, and so L/K is a Galois extension and $\operatorname{Gal}(L/K) \simeq S_n$.

(a) Prove that $A^{S_n} = F[\sigma_1, \dots, \sigma_n]$. (You are allowed to use without proof that $F[\sigma_1, \dots, \sigma_n]$ is isomorphic to the ring of polynomials in n variables.)

(b) Prove that $\operatorname{Spec}(F[x_1, \dots, x_n]) \to \operatorname{Spec}(F[\sigma_1, \dots, \sigma_n])$ has finite fibers.

8. For $\mathfrak{a} \trianglelefteq A$, let $\mathfrak{a}[x] := \{\sum_{i=0}^{m} a_i x^i | \, a_i \in \mathfrak{a}, m \in \mathbb{Z}^+\}$. Let $f : A \hookrightarrow A[x], f(a) := a$.

(a) Convince yourself that $\mathfrak{a}^e = \mathfrak{a}[x]$. Show that $\operatorname{Spec} A \overset{e}{\hookrightarrow} \operatorname{Spec}(A[x])$ is a well-defined injection.

(b) Prove that, if $\mathfrak{q}$ is a $\mathfrak{p}$-primary of A, then $\mathfrak{q}^e$ is a $\mathfrak{p}^e$-primary ideal of $A[x]$.

(c) Suppose k is a field. Prove that $0 \subsetneq \langle x_1 \rangle \subsetneq \cdots \subsetneq \langle x_1, \dots, x_n \rangle$ is a chain of prime ideals of $k[x_1, \dots, x_n]$ and $\langle x_1, \dots, x_r \rangle^m$ is $\langle x_1, \dots, x_r \rangle$-primary for any $1 \le r \le n$ and positive integer m.