# Besicovitch Covering

Friday, February 06, 2015
4:25 PM

Let $E$ be a bounded subset of $\mathbb{R}^N$. For any $x \in E$, let $B_x$ be an open ball centered at $x$. Then $\exists \{x_n\}_{n=1}^{\infty} \subseteq E$ s.t.

$$\mathbb{1}_E \le \sum_{n=1}^{\infty} \mathbb{1}_{B_{x_n}} \ll_N \mathbb{1}_E .$$

<u>Hint</u>. Proceed in a greedy way.

Choose $x_1 \in E$ s.t. the radius $r_1$ of $B_x$ is at least $\frac{3}{4} \sup_{x \in E} \{\text{radius of } B_x\}$. Now define $x_i$'s recursively.

$x_n \in E \setminus (B_{x_1} \cup \cdots \cup B_{x_{n-1}})$ s.t.

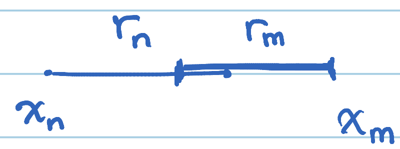
the radius $r_n$ of $B_{x_n} \ge \frac{3}{4} \underbrace{\sup_{x \in E \setminus (B_{x_1} \cup \cdots \cup B_{x_{n-1}})} (\text{of radius of } B_x)}_{\rho_n}$

---

$r_n \ge \frac{3}{4} \rho_n \ge \frac{3}{4} \rho_m \ge \frac{3}{4} r_m$ if $n \le m$

$r_n \quad r_m$

$d(x_n, x_m) \ge r_n > \frac{1}{3} r_n + \frac{2}{3} r_n$




$$d(x_n, x_m) \geq r_n \geq \frac{1}{3} r_n + \frac{2}{3} r_n$$
$$\geq \frac{1}{3} r_n + \frac{1}{2} r_m$$
$$> \frac{1}{3} r_n + \frac{1}{3} r_m.$$

$\Rightarrow \{B(x_n, \frac{1}{3} r_n)\}$ are disjoint.

$\overset{?}{\Rightarrow} r_n \longrightarrow 0.$ ( $E$ is bounded)

$\overset{?}{\Rightarrow} \{B_{x_n}\}$ is a covering.

---

For the other part, it is enough to prove that for any $k$

$B_{x_k}$ intersects at most $C(N)$ many of $B_{x_1}, \dots, B_{x_{k-1}}$.

We split them into two groups:

large constant

$$\text{"Small"}_k := \{ B_{x_i} \mid B_{x_i} \text{ intersects } B_{x_k};\ r_i \leq M\ r_k \}$$

$$\text{"Large"}_k := \{ B_{x_i} \mid B_{x_i} \text{ intersects } B_{x_k};\ r_i > M\ r_k \}.$$

---

$$\left.\begin{array}{l} x \in B(x_i, r_i/3) \\ B_{x_i} \in \text{"Small"} \end{array}\right\} \overset{?}{\Rightarrow} |x - x_k| \leq |x - x_i| + |x_i - x_k|$$
$$\leq r_i/3 + r_i + r_k$$
$$\leq (4/3\, M + 1) r_k.$$

$\Rightarrow \{ B(x_i, r_i/3) \mid B_{x_i} \in \text{"Small"} \}$ contains disjoint subdisks of $B(x_k, (4/3 M \quad +1) r_k)$.

$$\Rightarrow \left(\frac{4}{3}M \quad +1\right)^N r_k^N \geq \frac{1}{3^N} \sum_{B_{x_i} \in \text{"Small"}} r_i^N \geq \frac{1}{4^N} |\text{"Small"}| \, r_k^N$$

$$\Rightarrow 4^N \left(\frac{4}{3}M \quad +1\right)^N \quad \geq |\text{"Small"}| .$$

---

To control $|\text{"Large"}|$, we get a lower bound for $\widehat{x_i x_k x_j}$

for any $i \leq j \leq k$, by choosing $M$ large enough.

$i \leq j \Rightarrow |x_i - x_j| \geq r_i$

$B_{x_i}, B_{x_j}$ intersect $B_{x_k} \Rightarrow |x_i - x_k| \leq r_i + r_k$

and $|x_j - x_k| \leq r_j + r_k$

- $|x_i - x_k| \geq r_i$ and $|x_j - x_k| \geq r_j$

$$\Rightarrow \cos\theta \leq \frac{(r_i + r_k)^2 + (r_j + r_k)^2 - r_i^2}{2 r_i r_j}$$

$$= \frac{2 r_k^2 + 2 r_k (r_i + r_j) + r_j^2}{2 r_i r_j}$$

$$\leq \frac{1}{M^2} \quad + \frac{1}{M}\left(1 + \frac{4}{3}\right) + \frac{2}{3}$$

So, if $M$ is large enough, then $\theta \geq \theta_0 > 0$

$\Rightarrow |\text{"Large"}| \ll_N 1 .$