Exercise: Haar measure decomposition.

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9:21 AM

<u>Theorem</u>. Let G be a metrizable complete, locally compact topological group. Let G_1, G_2 be two <u>closed</u> subgroups of G.

Suppose (1) G_1G_2 is an open co-null subgroup.

(2) $H := G_1 \cap G_2$ is a compact subgroup.

Then, $\forall f \in C_c(G)$,

$$\int_G f(g)\, d\lambda_G(g) = \int_{G_1}\int_{G_2} f(g_1 g_2) \frac{\Delta_{G_2}(g_2)}{\Delta_G(g_2)}\, d\lambda_{G_2}(g_2)\, d\lambda_{G_1}(g_1)$$

<u>Hint</u>. (0) Let μ be the restriction of λ_G to G_1G_2.

(1)
$$G_1 \times G_2/\mathrm{diag}(H) \xrightarrow{\Phi} G_1G_2$$
$$[(g_1, g_2)] \longmapsto g_1 g_2^{-1}$$

is a homeomorphism. Let $\nu := \Phi^*(\mu)$.

(2) $\forall f \in C_c(G_1 \times G_2)$, $\overline{f}([(g_1, g_2)]) := \int_H f(g_1 h, g_2 h)\, d\lambda_H(h)$

is in $C_c(G_1 \times G_2/\mathrm{diag}(H))$.

(3)
$$C_c(G_1 \times G_2) \longrightarrow \mathbb{C}.$$
$$f \longmapsto \int \overline{f}\, d\nu.$$

$$f \longmapsto \int_{G_1 \times G_2/\mathrm{diag}\, H} \overline{f}\, d\upsilon .$$

is a positive functional $\Rightarrow \exists$ a Radon measure η

s.t. $$\int_{G_1\times G_2} f(g_1,g_2)\, d\eta = \int_{G_1\times G_2/\mathrm{diag}\, H} \int_H f(g_1 h, g_2 h)\, d\lambda_H(h)\, d\upsilon$$

$\boxed{\text{prob. Haar measure}}$

$\Rightarrow f \in C_c(G)$,

$$\int_{G_1\times G_2} f(g_1 g_2^{-1})\, d\eta = \int_{G_1 G_2} f(g_1 g_2^{-1})\, d\lambda_G = \int_G f(g)\, d\lambda_G(g).$$

Use the above for $L_{x_1} \circ R_{x_2}(f)$ to conclude

$$\int_G f(g)\, \Delta_G(x_2)\, d\lambda_G(g) = \int_{G_1\times G_2} f(x_1^{-1} g_1 g_2^{-1} x_2)\, d\eta(g_1,g_2)$$

④ We would like to compute $d\big((g_1,g_2)\cdot\eta\big)/d\eta$:

$\forall F \in C_c(G_1\times G_2)$, let $\overline{F}(g_1,g_2) = \int_H F(g_1 h, g_2 h)\, d\lambda_H(h)$

in $C_c(G_1\times G_2/\mathrm{diag}(H))$. Identify $G_1\times G_2/\mathrm{diag}(H)$ with $G_1 G_2$

to get a compactly supported continuous function $\widehat{F}$ on

$G_1 G_2$ (and so on G).

$$\int F(g_1,g_2)\, d((x_1,x_2)\cdot\eta)(g_1,g_2) =$$

$G_1 \times G_2$

$$\int_{G_1\times G_2} F(x_1^{-1}g_1, x_2^{-1}g_2)\, d\eta(g_1,g_2) =$$

$$\int_{G_1\times G_2/\mathrm{diag}(H)} \int_H F(x_1^{-1}g_1 h, x_2^{-1}g_2 h)\, d\lambda_H(h)\, d\nu([g_1,g_2]) =$$

$$\int_G (L_{x_1}\circ R_{x_2})(\hat{F})(g)\, d\lambda_G(g) =$$

$$\int_G \hat{F}(g)\, \Delta_G(x_2)\, d\lambda_G(g) =$$

$$\int_{G_1\times G_2} F(g_1,g_2)\, \Delta_G(x_2)\, d\eta(g_1,g_2).$$

$$\Rightarrow d((x_1,x_2)\cdot\eta)/d\eta = \Delta_G(x_2).$$

5 From the above equality conclude that

$$\Delta_G(g_2)\, d\eta(g_1,g_2) = d(\lambda_{G_1}\times\lambda_{G_2}).$$

$$\Rightarrow \int_G f(g)\, d\lambda_G(g) = \int_{G_1}\int_{G_2} f(g_1 g_2^{-1})\, \Delta_G(g_2)\, d\lambda_{G_2}(g_2)\, d\lambda_{G_1}(g_1).$$

6 Now we finish the proof by changing g_2^{-1} to g_2.

Let $\iota : G_2 \to G_2$, $\iota(g) := g^{-1}$. Then

$$\iota^* \lambda_{G_2} = \rho_{G_2} \Rightarrow d(\iota^*\lambda_{G_2})(g_2) = d\rho_{G_2}(g_2)$$
$$= \Delta_{G_2}(g_2)\; d\lambda_{G_2}(g_2)$$

$$\Rightarrow \int_G f(g)\, d\lambda_G(g) = \int_{G_1}\int_{G_2} f(g_1 g_2^{-1})\, \Delta_G(g_2)\; d\lambda_{G_2}(g_2)\, d\lambda_{G_1}(g_1)$$
$$= \int_{G_1}\int_{G_2} f(g_1 g_2)\, \Delta_G(g_2^{-1})\, \Delta_{G_2}(g_2)\; d\lambda_{G_2}(g_2)\, d\lambda_{G_1}(g_1)$$

(Theorem 8.32, A. Knapp, Lie groups beyond an introduction.)

We have already used the above formula to show

$$\rho(a)\; dn\; da\; dk$$

is a Haar measure of $SL_n(\mathbb{R})$.

Here is another way to look at the Haar measure of $SL_n(\mathbb{R})$.

- Let $G := GL_n(\mathbb{R})^\circ = \{g \in GL_n(\mathbb{R}) \mid \det(g) > 0\}$, $G_1 := \{a I_n \mid a \in \mathbb{R}^+\}$, $G_2 = SL_n(\mathbb{R})$. Prove:

$$\int_{GL_n(\mathbb{R})^\circ} f([x_{ij}])\, \frac{dx_{11}\cdots dx_{nn}}{(\det[x_{ij}])^n} = \int_{\mathbb{R}^+}\int_{SL_n(\mathbb{R})} f(rg)\, dg\, \frac{dr}{r}$$

$\mathbb{R}$' $SL_n(\mathbb{R})$

for some Haar measure dg of $SL_n(\mathbb{R})$.

Remark. Siegel, in "A Mean Value Theorem in Geometry of Numbers", Ann. of Math., 2^{nd} series, 46, no. 2, (1945) 340–347, uses the above Haar measure of $SL_n(\mathbb{R})$ to compute $\mathrm{vol}(SL_n(\mathbb{R})/SL_n(\mathbb{Z}))$. He showed that w.r.t. this measure the above volume is

$$\frac{1}{n}\,\zeta(2)\cdot\zeta(3)\cdot\cdots\cdot\zeta(n).$$