Howe-Moore for SL(n,R)

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<u>Thm</u>. $\pi: SL_n(\mathbb{R}) \to \mathcal{U}(\mathcal{H})$ continuouse unitary rep'n with no non-zero invariant vectors then

$$\forall v, w \in \mathcal{H} \setminus 0, \quad \langle \pi(g) v, w \rangle \xrightarrow[g \to \infty]{} 0 .$$

<u>First prove it for n=2.</u>

<u>Observation</u>. It is enough to show

$$\langle \pi\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} v, w \rangle \xrightarrow[a \to \infty]{} 0 .$$

(use KAK decomposition)

<u>Mautner Phenomena</u> We say (G,H) satisfy Mautner Phenomenon if

① $H \subseteq G$ is a subgroup.

② For any unitary rep'n $\pi: G \to \mathcal{U}(\mathcal{H})$ we have

$$\mathcal{H}^H = \mathcal{H}^G,$$

where $\mathcal{H}^X := \{ v \in \mathcal{H} \mid \forall g \in X, \ \pi(g)(v) = v \}$.

<u>Lemma</u>. Suppose $a_i, u \in G$ and $\pi: G \to \mathcal{U}(\mathcal{H})$ is a unitary rep'n. Suppose $a_i^{-1} u a_i \xrightarrow[i \to \infty]{} 1$ and $\pi(a_i) v = v$. Then

$$\pi(u) v = v .$$

<u>Pf</u>. $\langle \pi(u) v, v \rangle = \langle \pi(u) \pi(a_i) v, \pi(a_i) v \rangle$

$$= \langle \pi(a_i^{-1} u a_i) v, v \rangle$$

$$\longrightarrow \langle v, v\rangle$$

$$\Rightarrow \|\pi(u)v - v\|^2 = \|\pi(u)v\|^2 + \|v\|^2 - 2\,\mathrm{Re}\,\langle \pi(u)v, v\rangle$$
$$= 2\|v\|^2 - 2\|v\|^2 = 0.$$

$$\Rightarrow \pi(u)v = v. \qquad \blacksquare$$

<u>Corollary</u>. (AN, A) satisfies Mautner Phenomenon where
$$A = \left\{ \begin{bmatrix} a & \\ & a^{-1} \end{bmatrix} \middle| \ a \in \mathbb{R}^+ \right\} \text{ and } N = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \middle| \ x \in \mathbb{R} \right\}.$$

<u>Corollary</u> $(SL_2(\mathbb{R}), A)$ satisfies MP.

<u>Pf.</u> By the previous corollary (AN^+, A) and (AN^-, A) satisfy MP $\Rightarrow (\langle N^+, A, N^-\rangle, A)$ satisfies MP

$\Rightarrow (SL_2(\mathbb{R}), A)$ satisfies MP. $\blacksquare$

<u>Theorem</u> $(SL_2(\mathbb{R}), N^+)$ satisfies MP.

<u>Pf.</u> Let $v_0 \in \mathcal{H}^{N^+}$, and $f(g) := \langle \pi(g)v_0, v_0\rangle$. Then

$$\forall u_1, u_2 \in N^+, \quad f(u_1 g u_2) = \langle \pi(u_1 g u_2) v_0, v_0\rangle$$
$$= \langle \pi(g) v_0, v_0\rangle$$
$$= f(g).$$

So $f : SL_2(\mathbb{R}) \longrightarrow \mathbb{R}$ has the following properties:

① f is continuous.

② f is bi-N^+-invariant.

③ $|f| \leq \|v_0\|^2$.

So f induces continuous functions on $N\backslash SL_2(\mathbb{R})/N$.

Claim $SL_2(\mathbb{R})/N \longrightarrow \mathbb{R}^2\setminus\{0\}$ is a G-equivariant homeomorphism.

$$gN \longmapsto g\vec{e}_1$$

Pf of claim. $SL_2(\mathbb{R}) \curvearrowright \mathbb{R}^2$ and the stabilizer of $\vec{e}_1$ is $N \Rightarrow$ the above map is a bijection.

$g_iN \longrightarrow gN \iff \exists\, u_i \in N$ s.t. $g_iu_i \longrightarrow g$

$$\Longrightarrow g_i\vec{e}_1 = g_iu_i\vec{e}_1 \longrightarrow g\vec{e}_1.$$

$$g_i\vec{e}_1 \longrightarrow g\vec{e}_1 \Rightarrow g^{-1}g_i\vec{e}_1 \longrightarrow \vec{e}_1$$

$$\Rightarrow \underbrace{g^{-1}g_i}_{h_i} = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}, \quad a_i \longrightarrow 1, \; c_i \longrightarrow 0.$$

$$h_i\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_i & a_ix+b_i \\ c_i & c_ix+d_i \end{bmatrix} \Rightarrow h_i\begin{bmatrix} 1 & -\frac{b_i}{a_i} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_i & 0 \\ c_i & \frac{a_id_i-b_ic_i}{a_i} \end{bmatrix}$$

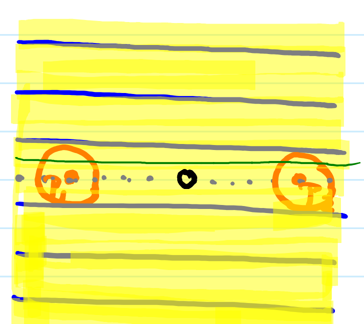
$\boxed{i \gg 1 \Rightarrow a_i \neq 0}$

$$= \begin{bmatrix} a_i & 0 \\ c_i & a_i^{-1} \end{bmatrix} \longrightarrow I$$

$$\Rightarrow g^{-1}g_iu_i \longrightarrow I \Rightarrow g_iu_i \longrightarrow g \Rightarrow g_iN \longrightarrow gN. \quad \blacksquare$$

So f induces a continuous function on $\mathbb{R}^2\setminus\{0\}$ which is N-invariant, i.e. $\bar{f}\left(\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} v\right) = \bar{f}(v)$ for any $a \in \mathbb{R}$ and $v \in \mathbb{R}^2\setminus\{0\}$.

So $\bar{f}$ is constant on N-orbits: lines parallel to the x-axis, but the x-axis, apriori.



$\forall\, p_1$ and p_2 on the x-axis $\setminus\{0\}$

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and any ε, $\exists\, \delta$ s.t.

$\forall\, q_1 \in B_\delta(p_1)$ and $q_2 \in B_\delta(p_2)$, $|f(p_i) - f(q_i)| < \varepsilon$.

Consider $y = \delta/2$. This line intersects $B_\delta(p_i)$. Let q_i's be on these intersections $\Longrightarrow \bar{f}(q_1) = \bar{f}(q_2)$

$$\Rightarrow |\bar{f}(p_1) - \bar{f}(p_2)| = |\bar{f}(p_1) - \bar{f}(q_1) + \bar{f}(q_2) - \bar{f}(p_2)|$$

$$\leq |\bar{f}(p_1) - \bar{f}(q_1)| + |\bar{f}(p_2) - \bar{f}(q_2)|$$

$$\leq 2\varepsilon.$$

$$\Rightarrow \bar{f}(p_1) = \bar{f}(p_2). \quad \circledast$$

So $f\left(\begin{bmatrix} a & \\ & a^{-1}\end{bmatrix}\right) = \bar{f}\left(\begin{bmatrix} a & \\ & a^{-1}\end{bmatrix}\vec{e}_1\right) = \bar{f}(a\vec{e}_1) \underset{\circledast}{=} \bar{f}(\vec{e}_1) = f(I) = \|\vec{v}_0\|^2$

$\Rightarrow \pi\left(\begin{bmatrix} a & \\ & a^{-1}\end{bmatrix}\right)v_0 = v_0 \Rightarrow$ by the previous corollary, $\forall g \in SL_2(\mathbb{R})$

$\pi(g)v_0 = v_0$. $\blacksquare$

<u>Remark</u>. Suppose (G, H) satisfies MP, and $G \curvearrowright X$, where (X, μ) is a finite measure space. Suppose μ is G-invariant. Then

$$\mu \text{ is } G\text{-ergodic} \Longleftrightarrow \mu \text{ is } H\text{-ergodic}.$$

<u>Pf of Howe-Moore for $SL_2(\mathbb{R})$.</u>

If not, $\exists\, g_i \in G$, $v_1, v_2 \in \mathcal{H}$ s.t.

① $g_i \to \infty$

② $\|v_1\| = \|v_2\| = 1$

③ $\langle \pi(g_i) v_1, v_2 \rangle \geq \varepsilon_0 > 0$

$g_i = k_i^{(1)} d_i k_i^{(2)}$ where $k_i^{(1)}, k_i^{(2)} \in SO_2(\mathbb{R})$ and $d_i = \begin{bmatrix} a_i & \\ & a_i^{-1} \end{bmatrix}$

and $a_i \longrightarrow +\infty$.

Going to a subseq, we can assume $k_i^{(1)} \longrightarrow k^{(1)}$ and $k_i^{(2)} \longrightarrow k^{(2)}$.

$$\Longrightarrow \quad \langle \pi(d_i)\, \pi(k_i^{(1)}) v_1 \,,\, \pi(k_i^{(2)})^{-1} v_2 \rangle \geq \varepsilon_0$$

$$\pi(k_i^{(j)})^{(-1)^{j-1}} v_j = \pi(k^{(j)})^{(-1)^{j-1}} v_j + x_{ij} \quad \text{and} \quad \|x_{ij}\| < \varepsilon \quad \text{if } i \gg_\varepsilon 1$$

$$\Longrightarrow \quad \varepsilon_0 \leq \langle \pi(d_i)\pi(k^{(1)}) v_1 + \pi(d_i) x_{i1} \,,\, \pi(k^{(2)})^{-1} v_2 + x_{i2} \rangle$$

$$\leq \langle \pi(d_i)\, \pi(k^{(1)}) v_1 \,,\, \pi(k^{(2)})^{-1} v_2 \rangle$$

$$+ \langle \pi(d_i)\, x_{i1} \,,\, \pi(k_i^{(2)})^{-1} v_2 \rangle$$

$$+ \langle \pi(d_i) \pi(k_i^{(1)}) v_1 \,,\, x_{i2} \rangle$$

$$\leq \langle \pi(d_i)\, w_1 \,,\, w_2 \rangle + 2\varepsilon \qquad \text{(Cauchy-Schwartz)}$$

where $w_1 = \pi(k^{(1)})\, v_1$

and $w_2 = \pi(k^{(2)})^{-1} v_2$

So, for $i \gg 1$, $\varepsilon_0/2 \leq \langle \pi(d_i)\, w_1 , w_2 \rangle$.

By Aleoglu's theorem, passing to a subseq. $\pi(d_i) w_1 \longrightarrow w_0$ weak*.

So $\varepsilon_0/2 \leq \langle w_0 , w_2 \rangle$, and so $0 < \|w_0\| \leq 1$.

<u>Claim</u>. w_0 is N^+-invariant.

<u>Pf</u>. $\forall u \in N^+$, $d_i^{-1} u d_i \longrightarrow I \Rightarrow \pi(d_i^{-1} u d_i) \longrightarrow \pi(I)$ in operator norm.

$\Rightarrow \forall x_1^{(i)}, x_2^{(i)}$ of bounded norms we have

$$| \langle \pi(d_i^{-1} u d_i)\, x_1^{(i)}, x_2^{(i)} \rangle - \langle x_1^{(i)}, x_2^{(i)} \rangle | < \varepsilon$$

$\Rightarrow \forall\, x_1, x_2$ of bounded norms we have

$$\left|\langle \pi(d_i^{-1} u d_i)\, x_1^{(i)}, x_2^{(i)}\rangle - \langle x_1^{(i)}, x_2^{(i)}\rangle\right| \le \varepsilon$$

if $i \gg_\varepsilon 1$ (independent of $x_1^{(i)}$ and $x_2^{(i)}$).

$$\Rightarrow \quad \langle \pi(u)\pi(d_i)\, x_1^{(i)}, \pi(d_i)\, x_2^{(i)}\rangle = \langle x_1^{(i)}, x_2^{(i)}\rangle + O(\varepsilon)$$

if $i \gg_\varepsilon 1$.

Let $x_1^{(i)} = \pi(d_i)^{-1} w_0$ and $x_2^{(i)} = w_1$.

$$\Rightarrow \quad \langle \pi(u) w_0, \pi(d_i) w_1\rangle = \langle \pi(d_i)^{-1} w_0, w_1\rangle + O(\varepsilon)$$

$$= \langle w_0, \pi(d_i) w_1\rangle + O(\varepsilon)$$

if $i \gg_\varepsilon 1$.

Since $\pi(d_i) w_1 \longrightarrow w_0$ in weak*-topology, we have

$$\langle \pi(u) w_0, w_0\rangle = \langle w_0, w_0\rangle + O(\varepsilon)$$

for any $\varepsilon > 0$. Therefore

$$\langle \pi(u) w_0, w_0\rangle = \langle w_0, w_0\rangle \Rightarrow \pi(u) w_0 = w_0. \quad ∎$$

Since $(SL_2(\mathbb{R}), N^+)$ satisfies MP, w_0 is $SL_2(\mathbb{R})$-invariant which is a contradiction. ∎

<u>Pf of Howe-Moore for $SL_n(\mathbb{R})$</u>

If not, as in the proof of $n=2$, $\exists\, d_i = \mathrm{diag}(a_i^{(1)}, \dots, a_i^{(n)})$, v_1, and v_2 s.t. $a_i^{(1)} \ge a_i^{(2)} \ge \dots \ge a_i^{(n)} > 0$, $\|v_1\| = \|v_2\| = 1$, and

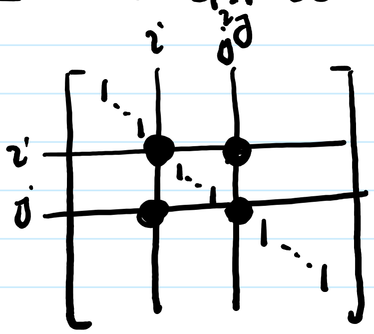
$$\langle \pi(d_i) v_1, v_2\rangle \ge \varepsilon_0 > 0,$$

and $d_i \longrightarrow \infty$. Since $d_i \longrightarrow \infty$, and $a_i \to \infty$. Since $d_i \to \infty$,

$$I := \{(i,j) \mid 1 \le i < j \le n,\ a_m^{(i)}/a_m^{(j)} \to \infty\} \neq \emptyset$$

Let ν_0 be a weak* limit of a subsequence of $\pi(d_i)v_1$.

As in the proof of $n=2$, ν_0 is invariant under $E_{ij} := \{I + x e_{ij} \mid x \in \mathbb{R}\}$ for any $(i,j) \in I$. Let G_{ij} be the subgp of $SL_n(\mathbb{R})$



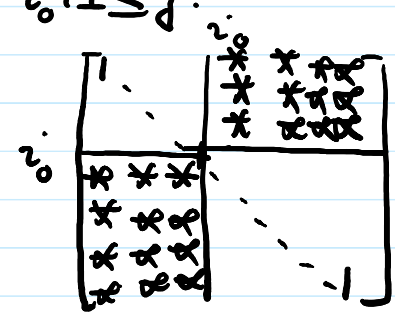
$G_{ij} \simeq SL_2(\mathbb{R})$. So (G_{ij}, E_{ij}) satisfies MP. Hence ν_0 is invariant under G_{ij} for any $(i,j) \in I$.

Notice that, for some i_0, $(i_0, i_0+1) \in I$, and this implies that $(i,j) \in I$ for any $i \le i_0$ and $i_0 + 1 \le j$.

$\Rightarrow$ ν_0 is E_{ji}-invariant.

$\Rightarrow$ ν_0 is invariant under

$\langle E_{ij}, E_{ji} \mid i \le i_0 < i_0+1 \le j \rangle$



These groups generate $SL_n(\mathbb{R})$. (why?) ∎

<u>Remark</u>. The general case is similar and is based on the fact that a semisimple group with no compact factors is generated by N^+, N^- where N^+ and N^- intersects all the simple factors and they

are unipotent radicals of opposite parabolics.

<u>Corollary</u>. Let G be an almost simple connected, non-compact Lie group. Let $\Gamma \subseteq G$ be a lattice, and $H \subseteq G$ be a non-compact subgp. Then the finite Haar measure of G/Γ is H-ergodic.

<u>Pf</u>. First notice that, if $f \in L^2(G/\Gamma)$ is G-invariant, then

$\exists$ a measurable function f_0 s.t. $f = f_0$ a.e. and f_0 is G-invariant. Since G acts transitively on G/Γ, f_0 is constant. So $L^2(\lambda_{G/\Gamma})^G = \mathbb{C}\,\mathbb{1}$. So the unitary rep'n of G on $L^2(\lambda_{G/\Gamma})^0$ has no non-zero invariant function.

Now suppose $f \in L^2(\lambda_{G/\Gamma})^0$ is H-invariant. Since H is NOT compact, $\exists \{h_n\} \subseteq H$, $h_n \to \infty$. Hence by Howe-Moore

$$\|f\|_2^2 = \langle \pi(h_n) f, f \rangle \longrightarrow 0$$

which is a contradiction. So any H-invariant function is constant, which implies $\lambda_{G/\Gamma}$ is H-ergodic. ∎