Arithmetic groups : basics.

Sunday, January 25, 2015
11:40 PM

Theorem $SL_n(\mathbb{Z})$ is a lattice in $SL_n(\mathbb{R})$.

Pf. Since $SL_n(\mathbb{R}) = [SL_n(\mathbb{R}), SL_n(\mathbb{R})]$, it is a unimodular group. By the reduction process we have proved that

$$SL_n(\mathbb{R}) = \mathfrak{S}_{2/\sqrt{3},\,1/2}\, SL_n(\mathbb{Z}).$$

Hence it is enough to prove

$$\lambda_{SL_n(\mathbb{R})}\left(\mathfrak{S}_{2/\sqrt{3},\,1/2}\right) < \infty.$$

We have proved that $d\lambda_{SL_n(\mathbb{R})} = \rho(a)\, dn\, da\, dk$.

$$\Rightarrow \lambda_{SL_n(\mathbb{R})}(\mathfrak{S}_{\alpha,\beta}) = \int_K \int_{A_\alpha} \int_{N_\beta} \rho(a)\, dn\, da\, dk$$

$$= \lambda_K(K)\, (2\beta)^{\frac{n(n-1)}{2}} \int_{A_\alpha} \rho(a)\, da.$$

$A \xrightarrow{\theta} (\mathbb{R}^+)^{n-1}$, $\mathrm{diag}(a_1,\dots,a_n) \mapsto (a_1 a_2^{-1}, a_2 a_3^{-1}, \dots, a_{n-1} a_n^{-1})$

- θ is an isomorphism (as topological groups).

$\rho(a) = \prod \alpha_i(a)^{m_i}$ where $m_i \in \mathbb{Z}^+$, $\alpha_i(a) = a_i a_{i+1}^{-1}$

$$\int \cdots \int^{\alpha} \int^{\alpha} x^{m_1} \cdots x^{m_{n-1}}\, dx_1 \cdots dx_{n-1}$$

$$\int_{A_\alpha} \rho(a)\, da = \int_0^\alpha \cdots \int_0^\alpha x_1^{m_1} \cdots x_{n-1}^{m_{n-1}} \frac{dx_1}{x_1} \cdots \frac{dx_{n-1}}{x_{n-1}}$$

$$= \prod_{i=1}^{n-1} \int_0^\alpha x_i^{m_i - 1}\, dx_i$$

$$= \prod_{i=1}^{n-1} \frac{\alpha^{m_i}}{m_i} < \infty. \qquad \blacksquare$$

Suppose $\mathbb{G}(\mathbb{R})$ is given by polynomial equations with coefficients in $\mathbb{Q}$. For example

① $\mathbb{SL}_n \longleftrightarrow \det X - 1 = 0$.

② For a given bilinear form f with coeff. in $\mathbb{Q}$

$$\mathbb{SO}_f \longleftrightarrow \det X - 1 = 0;$$
$$X \left[f(e_i, e_j) \right] X^T = \left[f(e_i, e_j) \right]$$

f can be symmetric $\rightsquigarrow$ quadratic form

or symplectic.

③ Let $k = \mathbb{Q}[\sqrt{D}]$ be a quadratic extension of $\mathbb{Q}$.

$h : (\mathbb{Q}[\sqrt{D}])^n \times (\mathbb{Q}[\sqrt{D}])^n \longrightarrow \mathbb{Q}[\sqrt{D}]$ a hermitian form

$\left(\Rightarrow \overline{h(e_j, e_i)} = h(e_i, e_j) \right)$

$\mathbb{SU} \longleftrightarrow$ ① $\det(X_1 + \sqrt{D}\, X_2) = 1$

$$SU_h \longleftrightarrow \text{① } \det(X_1+\sqrt{D}X_2)=1$$

$$\text{② } (X_1+\sqrt{D}X_2)\left[h(e_i,e_j)\right](X_1^T-\sqrt{D}X_2^T) = \left[h(e_i,e_j)\right].$$

Then we say G is algebraic and defined over $\mathbb{Q}$.

$\boxed{\text{Theorem (Borel-Harish-Chandra) Suppose } G \text{ is algebraic and defined over } \mathbb{Q}, \text{ and } \mathrm{Hom}(G^\circ, \mathbb{G}_m)_{\mathbb{Q}} = 1. \text{ Then } G(\mathbb{Z}) \text{ is a lattice in } G(\mathbb{R}).}$

Remark. It takes a lot more time to properly define algebraic groups. So we leave it to this minimum treatment for now.

- An important corollary of the above theorem is the following.

  Let k be a number field, $G/_k$ and $\mathrm{Hom}(G^\circ, \mathbb{G}_m)_k = 1$.

  Then $$G(\mathcal{O}_k) \hookrightarrow \prod_{\mathfrak{p}\in V_\infty(k)} G(k_{\mathfrak{p}})$$ is a lattice.

- The adelic version is also important which immediately gives you S-arithmetic groups:

$\mathrm{Hom}_k(G, \mathbb{G}_m) = 1 \implies G(k) \subseteq G(\mathbb{A}_k)$ is a lattice.