Space of unimodular lattices in $\mathbb{R}^n$, II.

Monday, January 12, 2015
4:39 PM

So far we have proved that

$$\begin{array}{ccc} GL_n(\mathbb{R})/GL_n(\mathbb{Z}) & \xrightarrow{\sim} & \Omega(\mathbb{R}^n) \\ \cup & & \cup \\ SL_n(\mathbb{R})/SL_n(\mathbb{Z}) & \xrightarrow{\sim} & \Omega^{(1)}(\mathbb{R}^n) \end{array}.$$

<u>Def.</u> $\forall \Lambda \in \Omega(\mathbb{R}^n)$, let

$$\delta(\Lambda) = \min\{\|v\| \mid v \in \Lambda \setminus \{0\}\}.$$

We start with the promised "reduction process" which gives us a nice $\mathbb{Z}$-basis for $\Lambda \in \Omega(\mathbb{R}^n)$.

<u>Lemma</u> Suppose $\Lambda \subseteq \mathbb{R}^n$ is a discrete subgroup, and $v_1, \dots, v_k \in \Lambda$. Let $V = \sum_{i=1}^{k} \mathbb{R} v_i$ and $V^\perp := \{x \in \mathbb{R}^n \mid x \perp V\}$. Then $\Pr_{V^\perp}(\Lambda)$ is a discrete subgroup of $V^\perp$.

<u>Pf</u>. Let $\mathcal{F} := \{\sum_{i=1}^{k} a_i v_i \mid |a_i| \le 1/2\}$. Then $\mathcal{F}$ is a compact subset of V, and $V = (\Lambda \cap V) + \mathcal{F}$.

Suppose to the contrary that $\Pr_{V^\perp}(\Lambda)$ is NOT discrete. Then $\exists\ \lambda_i \in \Lambda \setminus V$ s.t.

$$\Pr_{V^\perp}(\lambda_i) \longrightarrow 0.$$

For any i, $\exists\ \lambda_i' \in \Lambda \cap V$ s.t. $\Pr_V(\lambda_i) + \lambda_i' \in \mathcal{F}$

So going to a subseq. we can and will assume

$$\Pr_V(\lambda_i) + \lambda_i' \longrightarrow x \in \mathcal{F}.$$

Hence
$$\begin{aligned}\lambda_i + \lambda_i' &= \Pr_{V^\perp}(\lambda_i + \lambda_i') + \Pr_V(\lambda_i + \lambda_i') \\ &= \Pr_{V^\perp}(\lambda_i) + \left(\Pr_V(\lambda_i) + \lambda_i'\right) \\ &\xrightarrow[i\to\infty]{} x \in \mathcal{F} \subseteq V\end{aligned}$$

Since $\lambda_i + \lambda_i' \in \Lambda$ and Λ is discrete, we have

$$\lambda_i + \lambda_i' = x \in V \quad \text{if } i \gg 1$$

$\Rightarrow \lambda_i \in V$ which is a contradiction. ∎

Proposition (Reduction Process).

Let $\Lambda \subseteq \mathbb{R}^n$ be a discrete subgroup. Vectors $v_1, v_2, \ldots$ are defined recursively:

$v_1, v_2, \dots$ are defined recursively:

① $v_1 \in \Lambda$ s.t. $\|v_1\| = \delta(\Lambda)$, i.e. one of the shortest vectors of Λ.

② $v_{k+1} \in \Lambda$ s.t. $\|Pr_{V_k^\perp}(v_{k+1})\| = \delta(Pr_{V_k^\perp}(\Lambda))$, where $V_k = \sum_{i=1}^{k} \mathbb{R} v_i$. (Notice that it is well-defined, because of the previous lemma.)

Then we get finitely many vectors $v_1, \dots, v_m$;

$$\Lambda \subseteq V_m;$$

$$\Lambda \cap V_k = \bigoplus_{i=1}^{k} \mathbb{Z} v_i \quad \text{for} \quad 1 \le k \le m.$$

Pf. ② implies that $v_{k+1} \notin V_k$. So $\dim V_k = k$.

In particular we do not get more than $\underline{n}$ vectors.

Since by assumption the process stops after m steps, $Pr_{V_m^\perp}(\Lambda) = 0$. So $\Lambda \subseteq V_m$.

To prove the last part we proceed by induction and set $V_0 = 0$. So suppose $\Lambda \cap V_{k-1} = \bigoplus_{i=1}^{k-1} \mathbb{Z} v_i$, and let $\lambda \in \Lambda \cap V_k$. So $\exists\, n_i \in \mathbb{Z}$ and $\alpha_i \in \mathbb{R}$ s.t.

$i=1$

let $\lambda \in \Lambda \cap V_k$. So $\exists n_i \in \mathbb{Z}$ and $\alpha_i \in \mathbb{R}$ s.t.

$$\Lambda \ni \lambda - \sum_{i=1}^{k} n_i v_i = \sum_{i=1}^{k} \alpha_i v_i \quad \text{and}$$

$$|\alpha_i| \leq 1/2 .$$

Hence $\quad \mathrm{Pr}_{V_{k-1}^{\perp}}\left(\sum_{i=1}^{k} \alpha_i v_i\right) = \alpha_k \,\mathrm{Pr}_{V_{k-1}^{\perp}}(v_k) \in \mathrm{Pr}_{V_{k-1}^{\perp}}(\Lambda)$

Therefore $\quad \left\| \mathrm{Pr}_{V_{k-1}^{\perp}}\left(\sum_{i=1}^{k} \alpha_i v_i\right)\right\| \leq \frac{1}{2}\, \delta\left(\mathrm{Pr}_{V_{k-1}^{\perp}}(\Lambda)\right)$

$$\Rightarrow \mathrm{Pr}_{V_{k-1}^{\perp}}\left(\sum_{i=1}^{k} \alpha_i v_i\right) = 0 \Rightarrow \sum_{i=1}^{k} \alpha_i v_i \in \Lambda \cap V_{k-1} = \bigoplus_{i=1}^{k-1} \mathbb{Z} v_i$$

$\Rightarrow \alpha_i = 0 \quad$ as $\quad |\alpha_i| \leq 1/2 \Rightarrow$

$$\lambda = \sum_{i=1}^{k} n_i v_i \in \bigoplus_{i=1}^{k} \mathbb{Z}\, v_i . \qquad \blacksquare$$

Let me recall Gram-Schmidt process. Given linearly independent vectors $v_1, \dots, v_k \in \mathbb{R}^n$, we can get vectors $w_1, \dots, w_k$ that are pairwise orthogonal and for any $1 \leq i \leq k$, the $\mathbb{R}$-span V_i of $v_1, \dots, v_i = \bigoplus_{j=1}^{i} \mathbb{R} w_j$:

for any i, $\quad w_i = \mathrm{Pr}_{V_{i-1}^{\perp}} v_i$. So

For any i, $w_i = \Pr_{V_{i-1}^{\perp}} v_i$. So

$$v_1 = w_1$$

$$v_2 := w_2 + \frac{v_2 \cdot w_1}{w_1 \cdot w_1} w_1$$

$$v_3 := w_3 + \frac{v_3 \cdot w_2}{w_2 \cdot w_2} w_2 + \frac{v_3 \cdot w_1}{w_1 \cdot w_1} w_1$$

$$\vdots$$

$$v_k := w_k + \frac{v_k \cdot w_{k-1}}{w_{k-1} \cdot w_{k-1}} w_{k-1} + \cdots + \frac{v_k \cdot w_1}{w_1 \cdot w_1} w_1$$

$$\Rightarrow [v_1 \cdots v_k] = [w_1 \cdots w_k] \begin{bmatrix} 1 & \frac{v_2 \cdot w_1}{w_1 \cdot w_1} & \frac{v_3 \cdot w_1}{w_1 \cdot w_1} & \cdots & \frac{v_k \cdot w_1}{w_1 \cdot w_1} \\ & 1 & \frac{v_3 \cdot w_2}{w_2 \cdot w_2} & \cdots & \frac{v_k \cdot w_2}{w_2 w_2} \\ & & 1 & \cdots & \vdots \\ & & & & \\ & & & & 1 \end{bmatrix}$$

$$= \underbrace{[u_1 \cdots u_k]}_{\text{orthonormal columns}} \underbrace{\begin{bmatrix} \|w_1\| & & \\ & \ddots & \\ & & \|w_k\| \end{bmatrix}}_{\text{diagonal}} \underbrace{\begin{bmatrix} 1 & \frac{w_i \cdot v_j}{w_i \cdot w_i} \\ & \ddots \\ & & 1 \end{bmatrix}}_{\text{Upper triangular.}} . \quad (*)$$

Lemma. Suppose $v_1, \ldots, v_k \in \mathbb{R}^n$ are linearly independent, and $w_1, \ldots, w_k$ are the orthogonal vectors out of Gram–Schmidt process. Let

Schmidt process. Let $\mathcal{F}_k := \{ \sum_{i=1}^{k} \alpha_i w_i \mid |\alpha_i| \le 1/2 \}$.

Then $$\bigoplus_{i=1}^{k} \mathbb{Z} v_i + \mathcal{F}_k = \bigoplus_{i=1}^{k} \mathbb{R} v_i.$$

Pf. We proceed by induction on k.

$$x \in \bigoplus_{i=1}^{k} \mathbb{R} v_i \Rightarrow \exists!\ m \in \mathbb{Z} \text{ s.t.}$$

$$-1/2 \le \frac{x \cdot w_k}{w_k \cdot w_k} - m < 1/2.$$

$$\Rightarrow \quad x - m v_k = \Pr_{V_{k-1}} (x - m v_k) + \Pr_{V_{k-1}^{\perp}} (x - m v_k)$$

By the induction hypothesis, $\exists\ y \in \mathcal{F}_{k-1}$ and $\lambda \in \bigoplus_{i=1}^{k-1} \mathbb{Z} v_i$:

$$\Pr_{V_{k-1}} (x - m v_k) = \lambda + y.$$

$$\Rightarrow x - m v_k = \lambda + y + \frac{(x - m v_k) \cdot w_k}{w_k \cdot w_k} w_k$$

$$= \lambda + \underbrace{y + \left(\frac{x \cdot w_k}{w_k \cdot w_k} - m \right) w_k}_{\in \mathcal{F}_k}$$

$$\Rightarrow \quad x \in \bigoplus_{i=1}^{k} \mathbb{Z} v_i + \mathcal{F}_k. \qquad \blacksquare$$

Theorem. Let $\Lambda \in \Omega(\mathbb{R}^n)$. Then Λ has a $\mathbb{Z}$-basis

Theorem. Let $\Lambda \in \Omega(\mathbb{R}^n)$. Then Λ has a $\mathbb{Z}$-basis $v_1, \dots, v_n$ with the following properties:

Let $w_i = \mathrm{Pr}_{V_{i-1}^{\perp}}(v_i)$ where $V_{i-1} = \bigoplus_{j=1}^{i-1} \mathbb{R} v_j$.

$\Rightarrow$ ① $\|w_i\| / \|w_{i+1}\| \leq 2/\sqrt{3}$,

② $\left| \dfrac{v_i \cdot w_j}{w_j \cdot w_j} \right| \leq 1/2$, for $i > j$,

③ $\|w_1\| \cdot \|w_2\| \cdot \dots \cdot \|w_n\| = \mathrm{vol}(\mathbb{R}^n / \Omega)$.

Pf. We essentially use the reduction process. We notice that at each step we have lots of freedom in choosing v_k. At the moment our only restriction is

$$\|\mathrm{Pr}_{V_{k-1}^{\perp}}(v_k)\| = \delta(\mathrm{Pr}_{V_{k-1}^{\perp}}(\Lambda)).$$

In particular any vector in $v_k + \bigoplus_{i=1}^{k-1} \mathbb{Z} v_i$ can be chosen. By the above lemma, we can chose v_k s.t.

$$\mathrm{Pr}_{V_{k-1}}(v_k) \in \mathcal{F}_k.$$

So for any $1 < k < n$ and any $1 < i < k-1$ we have

$$\frac{1}{2} \geq \left| \frac{\mathrm{Pr}_{V_{k-1}}(v_k)\cdot w_j}{w_j \cdot w_j} \right| = \left| \frac{v_k \cdot w_j}{w_j \cdot w_j} \right| .$$

And $\|w_k\| = \|\mathrm{Pr}_{V_{k-1}^{\perp}}(v_k)\| = \delta(\mathrm{Pr}_{V_{k-1}^{\perp}}(\Lambda)) \leq \|\mathrm{Pr}_{V_{k-1}^{\perp}}(v_{k+1})\|$

$$\Rightarrow \|w_k\|^2 \leq \left\| \frac{w_k \cdot v_{k+1}}{w_k \cdot w_k} w_k + w_{k+1} \right\|$$

$$= \left| \frac{w_k \cdot v_{k+1}}{w_k \cdot w_k} \right|^2 \|w_k\|^2 + \|w_{k+1}\|^2$$

$$\leq \frac{1}{4} \|w_k\|^2 + \|w_{k+1}\|^2$$

$$\Rightarrow \|w_k\| / \|w_{k+1}\| \leq 2/\sqrt{3} .$$

By ⊛ above we have

$$\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \underbrace{K}_{\text{orthogonal matrix}} \begin{bmatrix} \|w_1\| & & \\ & \ddots & \\ & & \|w_n\| \end{bmatrix} \begin{bmatrix} 1 & & \frac{w_i \cdot v_j}{w_i \cdot w_i} \\ & \ddots & \\ & & 1 \end{bmatrix}$$

$$\Rightarrow \mathrm{vol}(\mathbb{R}^n/\Lambda) = |\det[v_1 \cdots v_n]| = \prod_{i=1}^{n} \|w_i\| .$$

■

Corollary ① $GL_n(\mathbb{R}) = \mathfrak{S}_{2/\sqrt{3},1/2}\, GL_n(\mathbb{Z})$

② $SL_n(\mathbb{R}) = \mathfrak{S}^{(1)}_{2/\sqrt{3},1/2}\, SL_n(\mathbb{Z})$

② $SL_n(\mathbb{R}) = \mathfrak{S}^{(1)}_{2/\sqrt{3},1/2}\, SL_n(\mathbb{Z})$

where $\mathfrak{S}_{a,b} := O(n) \cdot \left\{ \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \;\middle|\; d_i/d_{i+1} \le a,\ d_i \in \mathbb{R}^+ \right\}$

$\cdot \left\{ \begin{bmatrix} 1 & & n_{ij} \\ & \ddots & \\ & & 1 \end{bmatrix} \;\middle|\; |n_{ij}| \le b \right\}$,

and

$\mathfrak{S}^{(1)}_{a,b} := SO(n) \cdot \left\{ \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \;\middle|\; \begin{array}{l} d_i \in \mathbb{R}^+,\ d_i/d_{i+1} \le a, \\ d_1 \cdots d_n = 1 \end{array} \right\}$.

$\left\{ \begin{bmatrix} 1 & & n_{ij} \\ & \ddots & \\ & & 1 \end{bmatrix} \;\middle|\; |n_{ij}| \le b \right\}$.

Remark. $\mathfrak{S}_{a,b}$ and $\mathfrak{S}^{(1)}_{a,b}$ are called Siegel sets.

Pf of Corollary.

$\forall g_0 \in GL_n(\mathbb{R})$, $g_0\mathbb{Z}^n$ has a basis $v_1, \dots, v_n$ which

satisfies the properties of the above theorem $\Rightarrow$

$$g_0\mathbb{Z}^n = g\,\mathbb{Z}^n \quad \text{where } g = [v_1 \cdots v_n].$$

By the above theorem, $g \in \mathfrak{S}_{2/\sqrt{3},1/2}$.

$\Rightarrow g_0 \in \mathfrak{S}_{2/\sqrt{3},1/2}\, GL_n(\mathbb{Z})$.

$g_0 \in SL_n(\mathbb{R}) \Rightarrow g_0 = k\,a\,n\,\gamma$

$k \in O(n)$, $\gamma \in GL_n(\mathbb{Z})$ and $\underline{a}$ and $\underline{n}$

satisfy the desired properties.

satisfy the desired properties.

If $k \notin SO(n)$, then
$(\det(\gamma) = -1)$

$$g_0 = \underbrace{\underbrace{k\left(\begin{bmatrix} -1 & \\ & I \end{bmatrix}\right)}_{SO(n)} a \underbrace{\left(\begin{bmatrix} -1 & \\ & I \end{bmatrix} n \begin{bmatrix} -1 & \\ & I \end{bmatrix}\right)}}_{S^{(1)}_{2/\sqrt{3},\, 1/2}} \underbrace{\left(\begin{bmatrix} -1 & \\ & I \end{bmatrix} \gamma\right)}_{SL_n(\mathbb{Z})}$$

$\blacksquare$

Theorem (Mahler's compactness criteria)

$X \subseteq \Omega^{(1)}(\mathbb{R}^n)$ is precompact if and only if $\exists\, \delta_0 > 0$ s.t. $\forall\, \Lambda \in X$, $\delta(\Lambda) \geq \delta_0$.

Pf. ($\Longrightarrow$) It is easy to see that

$$\delta : \Omega^{(1)}(\mathbb{R}^n) \longrightarrow \mathbb{R}^+$$

is continuous. So $\delta(\overline{X})$ is a compact subset of $\mathbb{R}^+$. Hence $\inf \delta(\overline{X}) = \min \delta(\overline{X}) = \delta_0 > 0$.

($\Longleftarrow$) We know that $\Omega^{(1)}(\mathbb{R}^n)$ is homeomorphic to $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$:

$$\Lambda = g\mathbb{Z}^n \longrightarrow g\, SL_n(\mathbb{Z}),$$

and $g\, SL_n(\mathbb{Z}) = k(\Lambda)\, a(\Lambda)\, n(\Lambda)\, SL_n(\mathbb{Z})$

where $k(\Lambda) \in SO(n)$, $n(\Lambda) = \begin{bmatrix} 1 & n_{ij}(\Lambda) \\ & \ddots \end{bmatrix}$ s.t.

where $k(\Lambda) \in SO(n)$, $n(\Lambda) = \begin{bmatrix} 1 & & n_{ij}(\Lambda) \\ & \ddots & \\ & & 1 \end{bmatrix}$ s.t.

$|n_{ij}(\Lambda)| \leq 1/2$, and

$$a(\Lambda) = \begin{bmatrix} d_1(\Lambda) & & & \\ & d_2(\Lambda) & & \\ & & \ddots & \\ & & & d_n(\Lambda) \end{bmatrix} \text{ s.t.}$$

$d_1(\Lambda) = \delta(\Lambda)$, $\quad d_i(\Lambda)/d_{i+1}(\Lambda) \leq 2/\sqrt{3}$, and

$d_1(\Lambda)\, d_2(\Lambda) \cdots d_n(\Lambda) = 1$.

So $\{k(\Lambda) \mid \Lambda \in X\}$ and $\{n(\Lambda) \mid \Lambda \in X\}$ are clearly precompact.

$$1 \ll_{\delta_0, n} d_1(\Lambda) \left(\frac{\sqrt{3}}{2}\right)^i \leq d_{i+1}(\Lambda) \qquad \text{for any } 0 \leq i \leq n-1.$$

$$\Rightarrow \quad d_1(\Lambda)^{n-1} \left(\frac{\sqrt{3}}{2}\right)^{\frac{(n-1)(n-2)}{2}} d_n(\Lambda) \leq 1$$

$$\Rightarrow \quad d_n(\Lambda) \leq \left(\frac{1}{\delta_0}\right)^{n-1} \left(\frac{2}{\sqrt{3}}\right)^{\frac{(n-1)(n-2)}{2}} \ll_{\delta_0, n} 1.$$

$$\Rightarrow d_i(\Lambda) \leq \left(\frac{2}{\sqrt{3}}\right)^{n-i} d_n(\Lambda) \ll_{\delta_0, n} 1$$

$$\Rightarrow \forall i, \quad d_i(\Lambda) = \Theta_{\delta_0, n}(1)$$

So $\{a(\Lambda) \mid \Lambda \in X\}$ is precompact.

Hence $\{ k(\Lambda) a(\Lambda) n(\Lambda) SL_n(\mathbb{Z}) \mid \Lambda \in X \}$ is precompact

$\Rightarrow$ X is precompact. ■

Definition. For $\Lambda \in \Omega^{(1)}(\mathbb{R}^n)$ and $1 \leq i \leq n-1$, let

$$\delta_i(\Lambda) = \inf_{v_j \in \Lambda} \{ \mathrm{vol}(v_1, \ldots, v_i) \mid \mathrm{vol}(v_1, \ldots, v_i) \neq 0 \}.$$

Theorem. Suppose $g = k(g)\, a(g)\, n(g) \in \mathfrak{S}^{(1)}_{\alpha,\beta}$,

and $a(g) = \mathrm{diag}(a_1, a_2, \ldots, a_n)$.

Then $$a_1 \cdot a_2 \cdot \ldots \cdot a_k \ll_{\alpha,\beta} \alpha_k(g\mathbb{Z}^n) \leq a_1 a_2 \cdots a_k$$

Pf. $\mathrm{vol}(ge_1, ge_2, \ldots, ge_k)$

$= \mathrm{vol}(ane_1, ane_2, \ldots, ane_k)$

k is an isometry so it does NOT change volume.

$= \mathrm{vol}(ae_1, ae_2 + n_{12}\, ae_1, \ldots, ae_k + n_{1k}\, ae_1 + \cdots + n_{k-1\,k}\, ae_k)$

$= \mathrm{vol}(a_1 e_1, a_2 e_2, \ldots, a_k e_k) = \prod_{i=1}^{k} a_i$.

Lower bound:

Let us start with an easy observation: as in the proof of Mahler's compactness criteria:

$$a_i/a_{i+1} \le \alpha \implies a_1 \ll_\alpha a_i$$

$$\implies a_1 \cdots a_k \ll_\alpha a_{i_1} \cdots a_{i_k} \text{ for any } i_1 < \cdots < i_k.$$

Suppose $w_1', \dots, w_k'$ are linearly independent vectors in $g\mathbb{Z}^n$.

$\implies w_i' = g w_i$ where w_i's are linearly independent vectors in $\mathbb{Z}^n$

$$\implies \mathrm{vol}(w_1', \dots, w_k') = \mathrm{vol}(a n w_1, \dots, a n w_k)$$

$$\left|(ana^{-1})_{ij}\right| = \left|a_i a_j^{-1} n_{ij}\right| \ll_\alpha |n_{ij}| \ll_\beta 1.$$

$$\boxed{\begin{array}{l} a_i \ll_\alpha a_j \quad \text{if } i \le j \\ \implies a_i a_j^{-1} \ll_\alpha 1 \quad \text{if } i < j \end{array}}$$

$$\implies \mathrm{vol}(an w_1, \dots, an w_k) = \mathrm{vol}(ana^{-1} a w_1, \dots, ana^{-1} a w_k)$$

$$\ll_{\alpha,\beta} \quad \text{Contin. of vol \& boundedness of } ana^{-1}$$

$$\mathrm{vol}(a w_1, \dots, a w_k)$$

Notice that orthogonal projection onto a subspace does NOT increase the volume. (why?)

Since $w_1, \dots, w_k$ are linearly independent, k rows

$I = i_1, \dots, i_k$ of $[w_1 \cdots w_k]$ are linearly independent.

We project aw_j onto these components. So

$$\mathrm{vol}(aw_1, \dots, aw_k) \geq \mathrm{vol}(Pr_I(aw_1), \dots, Pr_I(aw_k))$$

$$= \mathrm{vol}\left(\begin{bmatrix} a_{i_1} & & \\ & \ddots & \\ & & a_{i_k} \end{bmatrix} Pr_I(w_1), \dots, \begin{bmatrix} a_{i_1} & & \\ & \ddots & \\ & & a_{i_k} \end{bmatrix} Pr_I(w_k) \right)$$

$$= \left| \det\left(\begin{bmatrix} a_{i_1} & & \\ & \ddots & \\ & & a_{i_k} \end{bmatrix} \right) \det\left[Pr_I(w_1) \cdots Pr_I(w_k) \right] \right|$$

$$\geq a_{i_1} \cdots a_{i_k} \gg_{\alpha} a_1 \cdots a_k \qquad \blacksquare$$

$$\boxed{\det\left[Pr_I(w_1) \cdots Pr_I(w_k) \right] \in \mathbb{Z} \setminus \{0\}}$$