# Exercise: Weyl's equidistribution.

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11:12 PM

Recall

**Thm** (Uniform convergence of Fourier expansion)

$$f \in C^\infty(\mathbb{R}/\mathbb{Z}) \Rightarrow \sum_{k=-N}^{N} \langle f, \chi_k \rangle \chi_k \xrightarrow[N\to\infty]{} f \quad \text{uniformly}$$

**Def.** We say $\{a_i\}_{i=1}^{\infty} \subseteq \mathbb{R}/\mathbb{Z}$ is equidistributed if

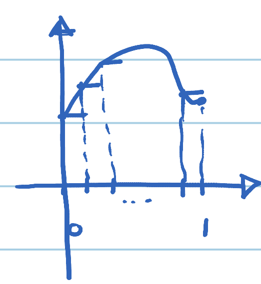
$$\forall\ 0 \le \alpha < \beta \le 1, \quad \frac{1}{N} \left| \{ n \in [0, N-1] \mid \alpha < a_n < \beta \} \right| \longrightarrow \beta - \alpha.$$

- Prove that $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{R}/\mathbb{Z}$ is equidistributed if and only if for any $f \in C^\infty(\mathbb{R}/\mathbb{Z})$

$$\frac{1}{N} \sum_{n=0}^{N-1} f(a_n) \xrightarrow[N\to\infty]{} \int_{S^1} f(t)\, dt$$

Hint. $f$ is absolutely continuous $\Rightarrow \forall \varepsilon, \exists M \gg_\varepsilon 1$ s.t.

if $|x-y| \le 1/M$, then $|f(x) - f(y)| < \varepsilon$.



Let $g(x) = f(i/M)$ if $i/M \le x < (i+1)/M$.

$\Rightarrow g$ is a step function and $\|f - g\|_\infty < \varepsilon$.

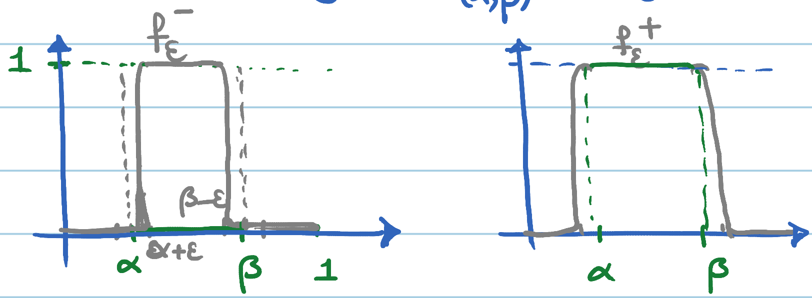
$$\Rightarrow \frac{1}{N} \sum_{n=0}^{N-1} f(a_n) = \frac{1}{N} \sum_{n=0}^{N-1} g(a_n) + O(\varepsilon)$$

$$\underset{N \gg_\varepsilon 1}{=} \langle g, \mathbb{1} \rangle + O(\varepsilon)$$

$$= \langle f, \mathbb{1} \rangle + O(\varepsilon).$$

For the other direction, $f_\varepsilon^- \leq I_{(\alpha,\beta)} \leq f_\varepsilon^+$

$f_\varepsilon^\pm \in C^\infty(\mathbb{R}/\mathbb{Z})$.




## Prove Weyl's Equidistribution Criteria:

$\{a_n\}_{n=1}^\infty \subseteq \mathbb{R}/\mathbb{Z}$ is equidistributed $\Leftrightarrow \forall k \in \mathbb{Z}\setminus\{0\}$,

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{2\pi i k a_n} \xrightarrow[N\to\infty]{} 0.$$

Hint. For any $f \in C^\infty(\mathbb{R}/\mathbb{Z})$, $\forall \varepsilon > 0$, if $m \gg_\varepsilon 1$, $\forall n$

$$\left| f(a_n) - \sum_{k=-m}^{m} \hat{f}(k)\, \chi_k(a_n) \right| \leq \varepsilon$$

$$\Rightarrow \quad \frac{1}{N}\sum_{n=0}^{N-1} f(a_n) = \sum_{k=-m}^{m} \hat{f}(k)\left(\frac{1}{N}\sum_{n=0}^{N-1}\chi_k(a_n)\right) + O(\varepsilon)$$

If $N \gg_{\varepsilon,m} 1$, $\left|\frac{1}{N}\sum_{n=0}^{N-1}\chi_k(a_n)\right| < \varepsilon/m$ for $k \neq 0$.

$$\Rightarrow \frac{1}{N}\sum_{n=0}^{N-1} f(a_n) = \hat{f}(0) + O_f(\varepsilon) = \int_{S^1} f(t)\,dt + O_f(\varepsilon)$$

$$\Rightarrow \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(a_n) = \int_{S^1} f(t)\,dt.$$

## Prove van der Corput's trick

Prove van der Corput's trick

Suppose $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{R}/\mathbb{Z}$. If $\forall h \in \mathbb{Z}^{>0}$ $\{a_{n+h} - a_n\}_{n=1}^{\infty}$ is equidistributed, then so is $\{a_n\}_{n=1}^{\infty}$.

Hint. By Weyl's criteria, we need to show

$$\frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i k a_n} \xrightarrow[N\to\infty]{} 0$$

if $k \neq 0$.

$\forall h \leq H \leq N$, (intermediate rang)

$$\frac{1}{N} \sum_{n=0}^{N-1} \chi_k(a_n) = \frac{1}{N} \sum_{n=0}^{N-1} \chi_k(a_{n+h}) + O\left(\frac{H}{N}\right)$$

$$\Rightarrow \left| \frac{1}{N} \sum_{n=0}^{N-1} \chi_k(a_n) \right| = \left| \frac{1}{N} \sum_{n=0}^{N-1} \left( \frac{1}{H} \sum_{h=0}^{H-1} \chi_k(a_{n+h}) \right) \right| + O\left(\frac{H}{N}\right)$$

Cauchy-Schwartz

$$\leq \left( \frac{1}{N} \sum_{n=0}^{N-1} \left| \frac{1}{H} \sum_{h=0}^{H-1} \chi_k(a_{n+h}) \right|^2 \right)^{1/2} + O\left(\frac{H}{N}\right)$$

$$= \left( \frac{1}{H^2} \sum_{h_1, h_2 = 0}^{H-1} \left( \frac{1}{N} \sum_{n=0}^{N-1} \chi_k(a_{n+h_1} - a_{n+h_2}) \right) \right)^{1/2} + O\left(\frac{H}{N}\right)$$

$$= \left( \frac{1}{H} + \frac{1}{H^2} \sum_{0 \leq h_1 \neq h_2 \leq H-1} \left( \frac{1}{N} \sum_{n=0}^{N-1} \chi_k(a_{n+h_1} - a_{n+h_2}) \right) \right)^{1/2}$$

$+ O\left(\frac{H}{N}\right)$

Fix H and let N go to infinity $\Rightarrow$ for any H

$$\overline{\lim_{N\to\infty}} \left| \frac{1}{N} \sum_{n=0}^{N-1} \chi_k(a_n) \right| \le 1/H .$$

Prove Weyl's theorem.

Let $P(x) \in \mathbb{R}[x] \setminus (\mathbb{Q}[x] \cup \mathbb{R})$. Then $\{P(n) + \mathbb{Z}\}_{n=0}^{\infty} \subseteq \mathbb{R}/\mathbb{Z}$ is equidistributed.

Hint. Use van der Corput's trick to prove this by induction on degree of $p$.

Remark. $\{\tau^n\}$, $n = 0, 1, 2, \dots$ is NOT equidistributed in $\mathbb{R}/\mathbb{Z}$, where $\tau = \frac{1+\sqrt{5}}{2}$. Why?

Hint. Use Fibonacci numbers.