Lecture 1: Objectives

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The main goal of this course is to go over Mostow's proof of Strong Rigidity for cocompact lattices in higher rank. Hopefully Eskin-Farb's proof of Quasi-Isometric Rigidity of such lattices will be discussed.

Along the way we will learn about symmetric spaces, Tits boundary, and a bit about other types of rigidity.

I will be mainly using Mostow's book. You can find more relevant references in the course webpage.

The first result on rigidity is due to Selberg:

Theorem (Selberg) A cocompact lattice in $SL_n(\mathbb{R})$, $n \geq 3$, is locally rigid.

Let's define a few terms:

Definition. Let G be a topological group. A subgroup Γ of G is called a lattice if

ⓐ Γ is discrete, ⓑ there is a regular, finite, G-invar.

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measure on G/Γ.

<u>Ex</u> ① $\mathbb{Z}^n$ is a lattice in $\mathbb{R}^n$.

② $SL_n(\mathbb{Z})$ is a lattice in $SL_n(\mathbb{R})$ (Minkowski's reduction theory) (due to Hurwitz)

③ Let M be a compact, orientable hyperbolic n-manifold.

Then its universal covering space is $\mathbb{H}^n$, and

$\pi_1(M) \simeq$ the group of Deck transformations

$\hookrightarrow \mathrm{Isom}(\mathbb{H}^n)^\circ$, as a discrete subgroup.

And $M \simeq \mathbb{H}^n / \pi_1(M)$.

$\mathrm{Isom}(\mathbb{H}^n)^\circ$ acts transitively on $\mathbb{H}^n$: for any $x_0 \in \mathbb{H}^n$, there is an isometry $\sigma_{x_0} : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ such that $\sigma_{x_0}(\ell(t)) = \ell(-t)$ for any parametrized geodesic ℓ which is at x_0 at $t=0$.

For any point x, let M be the middle point of the geodesic segment $x_0 x$. Then $(\sigma_M \circ \sigma_{x_0})(x_0) = x$

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and $\sigma_M \circ \sigma_{x_0}$ is orientation preserving.

One can show that, the set bijection between $\mathbb{H}^n$ and $G_{x_0} \backslash G$ where $G = \mathrm{Isom}(\mathbb{H}^n)^\circ$ and $G_{x_0} = \{g \in G \mid g \cdot x_0 = x_0\}$ is in fact a homeomorphism. Now we have

$$\left.\begin{array}{l} G_{x_0} \text{ is compact} \\ M \simeq \mathbb{H}^n / \pi_1(M) \end{array}\right\} \Longrightarrow \pi_1(M) \subseteq \mathrm{Isom}(\mathbb{H}^n)^\circ \text{ is a cocompact lattice.}$$

4 $SL_2(\mathbb{Z}[\sqrt{2}]) \subseteq SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$

$$a + \sqrt{2}\, b \longmapsto (a + \sqrt{2}\, b,\ a - \sqrt{2}\, b)$$

It is an (irreducible) lattice. (It is a corollary of Borel-Harish-Chandra's theorem.)

Lecture 1: Zariski topology (classical approach)

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To formulate Borel-Harish-Chandra's theorem, we briefly recall Zariski-topology.

<u>Zariski-topology</u> on $M_n(\mathbb{C})$: closed sets are common zeros of a family of polynomials in x_{ij} variables.

<u>Ex.</u> $SL_n(\mathbb{C})$ is a closed subset of $M_n(\mathbb{C})$.

<u>Ex.</u> $GL_n(\mathbb{C}) \hookrightarrow M_{n+1}(\mathbb{C})$, $x \mapsto \begin{bmatrix} x & \\ & \det(x)^{-1} \end{bmatrix}$ is a closed subset of $M_{n+1}(\mathbb{C})$.

<u>Def.</u> If V is the set of common zeros of a family of polynomials in T_i's with coefficients in a (char. 0) field k, we say V is defined over k.

- Notice that a Zariski-closed subgroup G of $GL_n(\mathbb{C})$ which is defined over k gives us a "family" of groups.

  For any unital commutative k-algebra A we get the group $G(A)$ of common zeros of those polynomials in A. In fact $A \longmapsto G(A)$ defines a functor from (unital commutative) k-algebras $\longrightarrow$ groups.

Borel–Harish-Chandra Suppose $G \subseteq GL_n(\mathbb{C})$ is a Zariski-closed subgroup, and $\mathrm{Hom}(G, \mathbb{C}^\times) = 1$.

Suppose G is common zeros of a family of polynomials with coefficients in $\mathbb{Q}$.

Then
$$G(\mathbb{Z}) := G \cap GL_n(\mathbb{Z}) \text{ is a lattice in } G(\mathbb{R}) := G \cap GL_n(\mathbb{R}).$$

Ex. $D := \mathbb{Q}[\sqrt{2}] \oplus \mathbb{Q}[\sqrt{2}]\, j$

- $j\,(a+\sqrt{2}b)\,j^{-1} = -\sqrt{2}\, b$
- $j^2 = 5$ (any number not in $N_{\mathbb{Q}[\sqrt{2}]/\mathbb{Q}}(\mathbb{Q}[\sqrt{2}])$).

$\Rightarrow D$ is a division algebra.

[There is a general way of constructing (cyclic) algebras:

L/K is a cyclic extenstion of degree n;

$\mathrm{Gal}(L/K) = \langle \sigma \rangle$;

$D = L \oplus Lx \oplus \cdots \oplus Lx^{n-1}$, $\forall \ell \in L$, $x^{-1} \ell x = \sigma(\ell)$

and $x^n = a \in K^\times$; Then D is a K-central simple algebra, and $D \simeq M_n(K) \Leftrightarrow a \in N_{L/K}(L^\times)$.]

Lecture 1: A Q-form of SL(2)

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D can be identified with $\left\{ \begin{bmatrix} X & Y \\ 5\overline{Y} & \overline{X} \end{bmatrix} \;\middle|\; X, Y \in \mathbb{Q}[\sqrt{2}] \right\}$

where $\overline{X}$ is the Galois conjugate of X. Let

$$SL_{1,D}(\mathbb{C}) := \left\{ \begin{bmatrix} x_1 + \sqrt{2}\,x_2 & y_1 + \sqrt{2}\,y_2 \\ 5(y_1 - \sqrt{2}\,y_2) & x_1 - \sqrt{2}\,x_2 \end{bmatrix} \;\middle|\; \begin{array}{l} x_1, x_2, y_1, y_2 \in \mathbb{C}, \\ \det(\cdot) \\ = x_1^2 - 2x_2^2 \\ -5(y_1^2 - 2y_2^2) = 1 \end{array} \right\}$$

It is easy to see that $SL_{1,D}(\mathbb{C}) = SL_2(\mathbb{C})$. So by Borel-Harish-Chandra, $SL_{1,D}(\mathbb{Z})$ is a lattice in $SL_{1,D}(\mathbb{R})$.

Since $\sqrt{2} \in \mathbb{R}$, we can see that $SL_{1,D}(\mathbb{R}) = SL_2(\mathbb{R})$.

Notice that $SL_{1,D}(\mathbb{Q}) \subseteq D$ and so for any $u \in SL_{1,D}(\mathbb{Q})$, $u - 1 \in D$ is invertible. Hence <u>no element of $SL_{1,D}(\mathbb{Q})$</u> has eigenvalue $=1$. In particular $SL_{1,D}(\mathbb{Q})$ has no unipotent element.

<u>Theorem</u>. (Borel-Harish-Chandra) $G \subseteq GL_n(\mathbb{C})$ common zeros of polynomials with coefficients in $\mathbb{Q}$; $\mathrm{Hom}(G, \mathbb{C}^\times) = 1$; $G(\mathbb{Q}) := G \cap GL_n(\mathbb{Q})$ has no unipotent element. Then $G(\mathbb{R})/G(\mathbb{Z})$ is compact.

Lecture 1: Borel-Harish-Chandra: number fields case

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Borel-Harish-Chandra's theorem implies the following version:

Let $G \subseteq GL_n(\mathbb{C})$ be Zariski-closed subgroup defined over a number field k. Suppose $\mathrm{Hom}(G, \mathbb{C}^\times) = 1$. Then $G(\mathcal{O}_k) := G \cap GL_n(\mathcal{O}_k)$ is a lattice in $\prod_{v \in V_\infty(k)} G(k_v)$ where $V_\infty(k)$ is the set of all archimedean places of k, and $\gamma \in G(\mathcal{O}_k)$ is sent to $(\sigma_1(\gamma), \ldots, \sigma_r(\gamma), \tau_1(\gamma), \ldots, \tau_s(\gamma))$ where $\sigma_i : k \to \mathbb{R}$ are the real embeddings and $\tau_i : k \to \mathbb{C}$ are the complex embeddings such that $\sigma_i \neq \sigma_j$ and $\tau_i \neq \tau_j$ and $\tau_i \neq \overline{\tau_j}$ for $i \neq j$.

<u>Ex.</u> Let $q(x_1, x_2, x_3) = x_1^2 + x_2^2 - \sqrt{2}\, x_3^2$, and $G = SO(q)$; i.e.

$$G := \{ g \in SL_3(\mathbb{C}) \mid q(g\vec{v}) = q(\vec{v}) \text{ for any } \vec{v} \in \mathbb{C}^3 \}.$$

So G is Zariski-closed subgroup defined over $\mathbb{Q}[\sqrt{2}]$.

Notice that over $\mathbb{R}$, the quadratic form q has signature $(2,1)$, and so $G(\mathbb{R}) \simeq SO(2,1) := \{ g \in SL_3(\mathbb{R}) \mid q_0(gv) = q_0(v) \}$ where $q_0(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2$. And one can see that $G(\mathbb{C})$

Lecture 1: Example of Borel-Harish-Chandra

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is a quotient of $SL_2(\mathbb{C})$ (to see this notice that

$sl_2(\mathbb{C}) := \{X \in M_2(\mathbb{C}) \mid tr(X)\}$ is a three dimensional space

and $SL_2(\mathbb{C}) \curvearrowright sl_2(\mathbb{C})$, $g \cdot X := gXg^{-1}$, $-\det(g\cdot X) = -\det(X)$

Notice $-\det\left(\begin{bmatrix} a & b \\ c & -a \end{bmatrix}\right) = +a^2 + bc$ is a quadratic form of sign. $(2,1)$.) So, by Borel-Harish-Chandra,

$$G(\mathbb{Z}[\sqrt{2}]) \hookrightarrow G(k_{\sigma_1}) \times G(k_{\sigma_2})$$

where $\sigma_1 : \mathbb{Q}[\sqrt{2}] \longrightarrow \mathbb{R}$ $\quad a+\sqrt{2}\,b \longmapsto a+\sqrt{2}\,b$,
and $\sigma_2 : \mathbb{Q}[\sqrt{2}] \longrightarrow \mathbb{R}$ $\quad a+\sqrt{2}\,b \longmapsto a-\sqrt{2}\,b$.

We have already said $G(k_{\sigma_1}) \simeq SO(2,1)$. Now

$g \in G(k_{\sigma_2}) \Rightarrow g$ preserves $\sigma_2(q)(x_1,x_2,x_3) = x_1^2 + x_2^2 + \sqrt{2}\,x_3^2$.

which has signature $(3,0)$. Hence $G(k_{\sigma_2}) \simeq SO(3)$.

Thus $G(\mathbb{Z}[\sqrt{2}]) \hookrightarrow SO(2,1) \times SO(3)$ is a lattice.

Since $SO(3)$ is compact, the projection of $G(\mathbb{Z}[\sqrt{2}])$ to $SO(2,1)$ gives us a lattice. Since a compact group does NOT have unipotent element, $G(\mathbb{Q}) \subseteq G(\mathbb{Q}_{\sigma_2})$ does not have unipotent element. Hence by Borel-Harish-Chandra $G(\mathbb{Z}[\sqrt{2}])$ is a cocompact lattice of $SO(2,1)$.

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In what extent the converse of these theorems hold?

One might think that, it should be possible to "perturb" the genertors of a lattice and get lattices with transcendental traces. This brings us to the next def.

<u>Definition</u>. We say a finitely generated subgroup Γ of G is <u>locally rigid</u> if $G\cdot\rho_0$ contains a nbhd of $\rho_0 \in \mathrm{Hom}(\Gamma, G)$, where

① $g\cdot\rho_0 : \Gamma \hookrightarrow G$, $(g\cdot\rho_0)(\gamma) = g\gamma g^{-1}$.

② $\mathrm{Hom}(\Gamma, G) = \{(g_1,\dots,g_n)\in G^n \mid r(g_1,\dots,g_n) = 1\}$, for any $r \in R$

where $\Gamma \simeq \langle x_1,\dots,x_n \mid R\rangle$.

<u>Remark</u>. $\mathrm{Hom}(\Gamma, G)$ can be viewed as an algebraic subvariety of $G\times\cdots\times G$.

<u>Lemma</u> Let $G \subseteq GL_n(\mathbb{C})$ be a Zariski-closed subgroup defined over $\overline{\mathbb{Q}}$. Let Γ be a finitely generated

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subgroup of $G(\mathbb{R})$. Suppose $\Gamma \subseteq G(\mathbb{R})$ is locally rigid. Then $\exists g \in G$ such $g\Gamma g^{-1} \subseteq GL_n(\overline{\mathbb{Q}})$.

<u>Lemma</u>. Let $V \subseteq \mathbb{C}^n$ be the set of common zeros of a family $\mathcal{F}$ of polynomials with coeff. in $\overline{\mathbb{Q}}$. Then $V(\overline{\mathbb{Q}})$ is dense in V in Archimedean topology.

<u>Proof</u>. Let $\mathfrak{a}$ be the radical of the ideal of $\overline{\mathbb{Q}}[T_1, \dots, T_n]$ which is generated by $\mathcal{F}$. As $\mathfrak{a} = \sqrt{\mathfrak{a}}$, there are prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ such that $\mathfrak{a} = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_m$. For any $(\alpha_1, \dots, \alpha_n) \in V$, $T_i \longmapsto \alpha_i$ gives us a $\overline{\mathbb{Q}}$-algebra homomorph. $\overline{\mathbb{Q}}[T_1, \dots, T_n] \xrightarrow{f_{\vec{\alpha}}} \mathbb{C}$; and $\mathfrak{a} \subseteq \ker(f_{\vec{\alpha}})$, and $\ker(f_{\vec{\alpha}})$ is a prime ideal of $\overline{\mathbb{Q}}[T_1, \dots, T_n]$. Hence, for some i, we have $\mathfrak{p}_i \subseteq \ker(f_{\vec{\alpha}})$, which means $\vec{\alpha}$ is a common zeros of elements of $\mathfrak{p}_i \supseteq \mathfrak{a} \supseteq \mathcal{F}$. Hence w.l.o.g. we can and will assume $A := \overline{\mathbb{Q}}[T_1, \dots, T_n]/\mathfrak{a}$ is an integral domain.

By the Noether normalization lemma, there are $\bar{y}_1, \dots, \bar{y}_d \in A$ which are algebraically independent and A is a finitely generated $\overline{\mathbb{Q}}[\bar{y}_1, \dots, \bar{y}_d]$-module.

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By the primitive element theorem $\exists\, a$ in the field of fractions F of A such that $F = \overline{\mathbb{Q}}(\bar{y}_1, \dots, \bar{y}_d)[a]$. Since F is a finite extension of $\overline{\mathbb{Q}}(\bar{y}_1, \dots, \bar{y}_d)$, we can and will assume $a \in A$. So $\exists\, q \in \overline{\mathbb{Q}}[\bar{y}_1, \dots, \bar{y}_d]$ such that $A[\frac{1}{q}] = \overline{\mathbb{Q}}[\bar{y}_1, \dots, \bar{y}_d][\frac{1}{q}][a]$. Moreover there is a polynomial $f(T; \vec{y}) \in \overline{\mathbb{Q}}[\bar{y}_1, \dots, \bar{y}_d][\frac{1}{q}]$ such that

$$\overline{\mathbb{Q}}[\bar{y}_1, \dots, \bar{y}_d][\tfrac{1}{q}][T] \big/ \langle f(T; \vec{y}) \rangle \xrightarrow{\sim} A[\tfrac{1}{q}]$$
$$T \longmapsto a.$$

Suppose $\vec{\alpha} \in V$ and $q(y_1(\vec{\alpha}), \dots, y_d(\vec{\alpha})) \neq 0$; then there are $y_i' \in \overline{\mathbb{Q}}$ such that y_i' is arbitrarily close to y_i. So we can assume $q(y_1', \dots, y_d') \neq 0$, and $f(T; \overline{y(\vec{\alpha})})$ is arbitrarily close to $f(T; \vec{y}')$. Hence one of the roots a' of $f(T; \vec{y}') = 0$ is arbitrarily close to a. Notice that any root of $f(T; \vec{y}') = 0$ is in $\overline{\mathbb{Q}}$. Hence we get $\vec{\alpha}' \in \overline{\mathbb{Q}}^n$ which is arbitrarily close to $\vec{\alpha}$ and $\vec{\alpha}' \in V$. Now using implicit function theorem, you can show $V \setminus \{\vec{\alpha} \in \mathbb{C}^n \mid q(\vec{y}(\alpha)) = 0\}$ is dense in Archimedean topology (?). ■

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Proof of Lemma regarding algebraic entries:

Let $X = \{(g_1, \dots, g_m) \in M_n(\mathbb{C}) \mid g_i \in G,\ r(g_1, \dots, g_m) = I \ \ \forall r \in R\}$ where $\Gamma \simeq F_m / \langle R \rangle$. So X is a variety defined over $\overline{\mathbb{Q}}$ and

$$\begin{aligned} \mathrm{Hom}(\Gamma, G) &\longrightarrow X \\ \phi &\longmapsto (\phi(\gamma_1), \dots, \phi(\gamma_m)) \end{aligned}$$

is a bijection where γ_i's are the image of generators of F_m under the isomorphism $F_m / \langle R \rangle \simeq \Gamma$. By the previous lemma $X(\overline{\mathbb{Q}})$ is dense in X with respect to the Archimedean topology. By the local rigidity assumption, $G \cdot \rho_0$ contains an Archimedean nbhd of X. Hence $\exists\, g \in G$ such that $g \cdot \rho_0 \in X(\overline{\mathbb{Q}})$, which implies $g \Gamma g^{-1} \subseteq GL_n(\overline{\mathbb{Q}})$. ■

<u>Remark</u>. In the above lemma, since Γ is f.g., there is a number field k such that $g \Gamma g^{-1} \subseteq GL_n(k)$.