Lecture 2: Borel's density theorem

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The main result which will be discussed today is Borel's density theorem. (We follow Dani's approach; the original argument in, Dani, A simple proof of Borel's density theorem, Math. Z. 174 (1980), was flawed.) Along the way we will learn about Poincaré recurrence theorem, Chevalley's theorem, and $SL_n(\mathbb{R}) = EL_n(\mathbb{R})$.

<u>Borel's density theorem</u> ("special case")

Let $G \subseteq GL_n(\mathbb{C})$ be a Zariski-closed subgroup defined over $\mathbb{R}$. Suppose $G(\mathbb{R})$ is generated by its 1-parameter unipotent subgroups. Then any lattice Γ in $G(\mathbb{R})$ is Zariski-dense.

Before we start the proof of Borel's density theorem, let's say what we mean by a unipotent flow, and see how restrictive the mentioned condition is.

<u>Definition</u>. A matrix $u \in GL_n(\mathbb{C})$ is called unipotent if

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all the eigenvalues of u are 1. Hence by Cayley–Hamilton's theorem $(u-I)^n=0$. And its Jordan blocks are of the form

$$J_{n_i}=\begin{bmatrix}1 & 1 & & 0\\ & \ddots & \ddots & \\ & & \ddots & 1\\ 0 & & & 1\end{bmatrix}=I+X_{n_i} \quad \text{where } X_{n_i}=\begin{bmatrix}0 & 1 & & 0\\ & \ddots & \ddots & \\ & & \ddots & 1\\ 0 & & & 0\end{bmatrix}.$$

Notice that
$$X_{n_i}^k=\begin{bmatrix}0 & \cdots & 0 & 1 & & 0\\ & & & \ddots & \ddots & \\ & & & & \ddots & 1\\ & & & & & 0\\ & & & & & \vdots\\ 0 & & & & & 0\end{bmatrix} \quad \text{and} \quad X_{n_i}^{n_i}=0.$$

Logarithmic and exponential functions.

Consider the formal power series:

$$\exp(t):=1+\frac{t}{1!}+\frac{t^2}{2!}+\cdots \quad \text{and} \quad \log(1-t):=t-\frac{t^2}{2}+\frac{t^3}{3}-\cdots$$

Then $\exp(\log t)=\log(\exp t)=t$ (as elements of $\mathbb{C}[\![t]\!]$)

Let $M_n(\mathbb{C})^{(n)}:=\{X\in M_n(\mathbb{C}) \mid X \text{ is nilpotent}\}$

$$=\{X\in M_n(\mathbb{C}) \mid X^n=0\} \text{ and}$$

$M_n(\mathbb{C})^{(u)}:=\{U\in M_n(\mathbb{C}) \mid U \text{ is unipotent}\}$

$$=\{U\in M_n(\mathbb{C}) \mid (U-I)^n=0\}.$$ So $M_n(\mathbb{C})^{(n)}$ and $M_n(\mathbb{C})^{(u)}$ are Zariski-closed sets which are defined over $\mathbb{Q}$.

For any $X\in M_n(\mathbb{C})^{(n)}$, $\exp(X)$ is well-defined (and a poly-map). And for any $U\in M_n(\mathbb{C})^{(u)}$, $\log U$ is well-defined (and a

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a polynomial map).

For $X = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$ we have

$$\exp(tX) = I + \frac{tX}{1!} + \frac{t^2X^2}{2!} + \cdots + \frac{t^{n-1}X^{n-1}}{(n-1)!}$$

$$= \begin{bmatrix} 1 & t/1! & t^2/2! & \cdots & t^{n-1}/(n-1)! \\ & 1 & t/1! & \cdots & t^{n-2}/(n-2)! \\ & & \ddots & \ddots & \vdots \\ & 0 & & & t/1! \\ & & & & 1 \end{bmatrix} \in M_n(\mathbb{C})^{(u)}.$$

So using Jordan form of a nilpotent matrix we get that $\exp : M_n(\mathbb{C})^{(n)} \longrightarrow M_n(\mathbb{C})^{(u)}$.

For $U = I + X$ where $X = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$ we have

$$\log U = -X + \frac{X^2}{2} - \frac{X^3}{3} + \cdots + (-1)^n \frac{X^{n-1}}{n-1}$$

$$= \begin{bmatrix} 0 & -1 & \frac{1}{2} & \cdots & \frac{(-1)^n}{n-1} \\ & 0 & -1 & \cdots & \vdots \\ & & \ddots & \ddots & \vdots \\ & 0 & & & -1 \\ & & & & 0 \end{bmatrix} \in M_n(\mathbb{C})^{(u)}.$$ Hence again using Jordan forms we get $\log : M_n(\mathbb{C})^{(u)} \longrightarrow M_n(\mathbb{C})^{(n)}$. And they define morphisms between these Zariski-closed sets. Moreover $t \mapsto \exp(tX)$ defines a unipotent flow if X is nilpotent.

Lecture 2: Unipotent flow; groups generated by them

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For $u \in M_n(\mathbb{C})^{(u)}$ we define $u^t := \exp(t \log u)$.

Let's compute u^m when $m \in \mathbb{Z}^+$ and $u = I + \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{bmatrix}$:

$$u^m = (I+X)^m = \sum_{i=0}^{m} \binom{m}{i} X^i = \begin{bmatrix} 1 & \binom{m}{1} & \cdots & \binom{m}{n-1} \\ & 1 & \ddots & \vdots \\ & & \ddots & \binom{m}{1} \\ & & & 1 \end{bmatrix}.$$

So in terms of $\underline{m}$ all the entries are rational polynomials of $\deg. \le n-1$.

<u>Kneser-Tits conjecture</u> G: semisimple, simply-connected F-group; G is almost F-simple, i.e. $G(F)/Z(G(F))$ is simple, and F-isotropic, i.e. $G(F)$ has (good) unipotent elements. Let $G(F)^+$ be the subgroup of $G(F)$ which is generated by (good) unipotent subgroups. Then

$$G(F) = G(F)^+.$$

Platonov proved this conjecture for local fields and gave a counter-example for the general case. The case of $F = \mathbb{R}$ was proved by E. Cartan. This implies: (show why?)

If G is a non-compact simple Lie group, then $G = G^+$.

Lecture 2: SL(n,R) is generated by its unipotents

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By the previous remark, you can see that the given condition in Borel's density theorem is NOT restrictive. Here we show

<u>Lemma</u>. $SL_n(F)$ is generated by its unipotent subgroups for any field F.

<u>Proof</u>. Let $EL_n(F) = \langle e_{ij}(t) \mid t \in F,\ i \neq j \rangle$ where

$$e_{ij}(t) = \begin{array}{cc} \begin{bmatrix} 1 & & t & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} & i \\ \quad\quad\ \ j & \end{array}$$

We use reduced row/column method:

Notice that $e_{ij}(t) \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_i + t v_j \\ \vdots \\ v_n \end{bmatrix}$.

$\forall g \in SL_n(F)$, $\exists 1 \leq i_0 \leq n$: $g_{i_0 1} \neq 0$. If $g_{11} = 0$, then the 11-entry of $e_{1 i_0}(1)\, g$ is non-zero. So after repeated use of reduced row method we get that

$$EL_n(F)\, g = EL_n(F) \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ & \ddots & & \vdots \\ 0 & & \ddots & \vdots \\ & & & g_{nn} \end{bmatrix}.$$

Now by a similar argument and using reduced column method we get $EL_n(F)\, g\, EL_n(F) = EL_n(F) \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} EL_n(F)$.

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To finish the proof it is enough to show (why ?)

$$EL_2(F)\begin{bmatrix} a_1 & \\ & a_2 \end{bmatrix}EL_2(F) = EL_2(F)\begin{bmatrix} 1 & \\ & a_1a_2 \end{bmatrix}EL_2(F)$$

To show this we use reduced row/column method:

$$\begin{bmatrix} a_1 & \\ & a_2 \end{bmatrix} \xrightarrow[\text{II}+a_1^{-1}\text{I}]{R} \begin{bmatrix} a_1 & \\ 1 & a_2 \end{bmatrix} \xrightarrow[\text{I}-a_1\text{II}]{R} \begin{bmatrix} 0 & -a_1a_2 \\ 1 & a_2 \end{bmatrix}$$

$$\xrightarrow[\text{II}-a_2\text{I}]{C} \begin{bmatrix} 0 & -a_1a_2 \\ 1 & 0 \end{bmatrix} \xrightarrow[\text{I}+\text{II}]{R} \begin{bmatrix} 1 & -a_1a_2 \\ 1 & 0 \end{bmatrix}$$

$$\xrightarrow[\text{II}+a_1a_2\text{I}]{C} \begin{bmatrix} 1 & 0 \\ 1 & a_1a_2 \end{bmatrix} \xrightarrow[\text{II}-\text{I}]{R} \begin{bmatrix} 1 & 0 \\ 0 & a_1a_2 \end{bmatrix}. \blacksquare$$

Remark. The above argument can be used to define Dioudonné determinant $\det : GL_n(D) \longrightarrow D^\times/[D^\times, D^\times]$ where D is a division algebra.

Lecture 2: Proof of Borel's density theorem

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Let H be the Zariski-closure of Γ in G.

<u>Lemma</u> H is a subgroup of G

<u>Proof</u> • $\forall \gamma\in\Gamma$, $\left.\begin{array}{l}\Gamma\subseteq\gamma H\\ \gamma H \text{ is closed}\end{array}\right\}\Rightarrow H\subseteq\gamma H.$

$\forall \gamma\in\Gamma$, $\left.\begin{array}{l}H\subseteq\gamma H\\ H\subseteq\gamma^{-1}H\end{array}\right\}\Rightarrow H=\gamma H.$

So $H=\Gamma H$.

• $\forall h\in H$, $h\Gamma\subseteq H \Rightarrow \overline{h\Gamma}\subseteq H \Rightarrow hH\subseteq\overline{h\Gamma}\subseteq H$

$\Rightarrow HH=H$.

• $\Gamma\subseteq H \Rightarrow \Gamma\subseteq H^{-1} \Rightarrow H\subseteq H^{-1}\Rightarrow H^{-1}\subseteq H$

$\Rightarrow H=H^{-1}$. ∎

By Chevalley's theorem, $\exists \rho: G\to GL_{m_0}(\mathbb{C})$ (with poly. maps with coefficients in $\mathbb{R}$) and $v_0\in\mathbb{R}^{m_0}\setminus\{0\}$ such that

$H=\{g\in G \mid \rho(g)[v_0]=[v_0]\}$ where $[v_0]\in\mathbb{P}(\mathbb{C}^{m_0})$.

So $G/H\to\mathbb{P}(\mathbb{C}^m)$, $gH\mapsto\rho(g)[v_0]$ is well-defined G-equivariant map.

Lecture 2: Action of a unipotent on the projective space

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Suppose to the contrary that $G(\mathbb{R}) \neq H(\mathbb{R})$. Since by the assumption $G(\mathbb{R})$ is generated by unipotents, $\exists\, u \in G(\mathbb{R})$ such that $\rho(u)[v_0] \neq [v_0]$. From the theory of algebraic groups $\rho(u)$ is unipotent. So we need to understand how a unipotent U acts on $P(\mathbb{C}^{m_0})$. The Jordan form of U is

$$\begin{bmatrix} J_{n_1} & & \\ & \ddots & \\ & & J_{n_k} \end{bmatrix} \text{ where } J_{n_i} = \begin{bmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{bmatrix}; \text{ so } U^m = \begin{bmatrix} J_{n_1}^m & & \\ & \ddots & \\ & & J_{n_k}^m \end{bmatrix}$$

And we have seen $J_{n_i}^m = \begin{bmatrix} 1 & \binom{m}{1} & \cdots & \binom{m}{n_i-1} \\ & 1 & \ddots & \vdots \\ & & \ddots & \binom{m}{1} \\ & & & 1 \end{bmatrix}$.

So
$$J_{n_i}^m \begin{bmatrix} c_0 \\ \vdots \\ c_{n_i-1} \end{bmatrix} = \begin{bmatrix} c_0 + c_1\binom{m}{1} + \cdots + c_{n_i-1}\binom{m}{n_i-1} \\ c_1 + c_2\binom{m}{1} + \cdots + c_{n_i-1}\binom{m}{n_i-2} \\ \vdots \\ c_{n_i-1} \end{bmatrix} = \begin{bmatrix} p_1(m) \\ \vdots \\ p_{n_i-1}(m) \end{bmatrix}$$

where $p_i(x) \in \mathbb{Q}[x]$. If j_0 is the largest index such that $c_{j_0} \neq 0$, then $\deg p_1 = j_0$, $\deg p_2 = j_0 - 1$, $\ldots$, $\deg p_{j_0+1} = 0$, $p_k = 0$ for $k > j_0+1$.

Hence $[p_1(m) : p_2(m) : \cdots : p_{n_i-1}(m)] \xrightarrow[m\to\infty]{} [1 : 0 : \cdots : 0]$ in $P(\mathbb{C}^{n_i})$.

So $J_{n_i}^m [v] \xrightarrow[m\to\infty]{} [e_1] \in P(\ker(J_{n_i} - I))$.

Therefore for a unipotent $U \in GL_{m_0}(\mathbb{C})$ we have

$$U^m[v] \xrightarrow[m\to\infty]{} [\ell_v] \in P(\ker(U-I)) \quad (\text{fixed points of } U \text{ in } P(\mathbb{C}^{m_0}).)$$

<u>Corollary</u>. Suppose U is a non-trivial unipotent element of $GL_{m_0}(\mathbb{C})$. Then for any $[v] \in P(\mathbb{C}^{m_0}) \setminus P(\ker(U-I))$ there is a nbhd $\mathcal{O}$ of $[v]$ such that for any $[w] \in \mathcal{O}$ we have

$$|\{n \in \mathbb{Z}^+ \mid U^n[w] \in \mathcal{O}\}| < \infty .$$

<u>Proof</u>. Since $P(\ker(U-I))$ is a close set, there is a nbhd $\mathcal{O}$ of $[v]$ whose closure does NOT intersect $P(\ker(U-I))$.

For any $[w] \in \mathcal{O}$, $\lim_{m\to\infty} U^m[w] = [\ell_w] \in P(\ker(U-I)) \setminus \overline{\mathcal{O}}$.

So, for large enough m, $U^m[w] \in P(\mathbb{C}^{m_0}) \setminus \overline{\mathcal{O}}$. Therefore

$$|\{n \in \mathbb{Z}^+ \mid U^n[w] \in \mathcal{O}\}| < \infty . \qquad \blacksquare$$

<u>Definition</u>. Let X be a topological space and $T: X \to X$ is a continuous map. We say $x \in X$ is a recurrent point of T if x is a limit point of the seq. $\{T^n x\}_{n=1}^{\infty}$. That means for any nbhd $\mathcal{O}$ of x, $|\{n \in \mathbb{Z}^+ \mid T^n x \in \mathcal{O}\}| = \infty$.

Lecture 2: Recurrent (projective) points of a unipotent; Poincare recurrence theorem.

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<u>Corollary</u>. Let U be a non-trivial unipotent matrix in $GL_{m_0}(\mathbb{C})$.

Then $[x] \in P(\mathbb{C}^{m_0})$ is U-recurrent if and only if

$[x] \in P(\ker(U-I))$, i.e. $[x]$ is fixed by U.

So far we have used only algebraic in put. Now we are going to use a little bit of dynamical systems.

<u>Theorem</u> (Poincaré recurrence theorem)

Let X be a locally compact, second countable, Hausdorff space. Let $T: X \longrightarrow X$ be a continuous bijection.

Let μ be a regular, finite, T-invariant measure on X. Then

(i) For any measurable subset E, for a.e. $x \in E$ we have $|\{n \in \mathbb{Z}^+ \mid T^n x \in E\}| = \infty$.

(ii) Almost every $x \in X$ is T-recurrent.

<u>Proof</u>. Let $r(E) := \{x \in E \mid \exists\, n \in \mathbb{Z}^+, T^n x \in E\}$; it means those points that go back to E at least once.

Lecture 2: Proof of Poincare recurrence theorem

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Then $r(E)$ is a measurable set (why?) and sets $T^n(E\setminus r(E))$ are pairwise disjoint: if, for $n>m$,

$$T^n(E\setminus r(E))\cap T^m(E\setminus r(E))\neq\emptyset, \text{ then}$$

$$\big(E\setminus r(E)\big)\cap T^{n-m}(E)\neq\emptyset \text{ which}$$

contradicts the fact that $E\cap T^{n-m}(E)\subseteq r(E)$.

Since μ is T-invariant, countabily additive and finite, we get $\mu(E\setminus r(E))=0$.

Therefore, for any $i\in\mathbb{Z}^{\geq 0}$, $\mu(r^i(E)\setminus r^{(i+1)}(E))=0$. And so $\mu(E\setminus\bigcap_{i=1}^{\infty} r^i(E))=\sum_{i=1}^{\infty}\mu(r^{i-1}(E)\setminus r^i(E))=0$, and

$$x\in\bigcap_{i=1}^{\infty} r^i(E)\iff |\{n\in\mathbb{Z}^+ \mid T^n x\}|=\infty .$$

(ii) Because of the assumptions on X, we get that X is metrizable. For $\delta>0$, let $\{B_i\}_{i=1}^{\infty}$ be a covering of X with balls of diameter $\leq\delta$. Let $X(\delta):=\bigcup_{i=1}^{\infty} r(B_i)$. Then by part (i), $\mu(X\setminus X(\delta))=0$. And, $\forall\, x\in X(\delta)$, $\exists\, i$ s.t. $x\in r(B_i)$ which means $\exists\, n_{x,\delta}\in\mathbb{Z}^+$ s.t. $T^{n_{x,\delta}}x\in B_i$. Therefore

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$d(T^{n_{x,\delta}} x, x) \le \delta$.

Hence $\mu(X \setminus \bigcap_{k=1}^{\infty} X(1/k)) = 0$ and for $x \in \bigcap_{k=1}^{\infty} X(1/k)$ we have that

$0 < n_{x,1} \le n_{x,2} \le \cdots$ s.t. $d(T^{n_{x,i}} x, x) \le 1/i$ and so x is T-recurrent. ■

Proof of Borel's density theorem.

Let H be the Zariski-closure of Γ. Then by Chevalley's theorem $\exists \rho: G \to GL_{m_0}(\mathbb{C})$ and $v_0 \in \mathbb{C}^{m_0} \setminus \{0\}$ s.t. $H = \{g \in G \mid \rho(g)[v_0] = [v_0]\}$. Since $G(\mathbb{R})$ is generated by its unipotent elements, $\exists\, u \in G$ s.t. $\rho(u)[v_0] \ne [v_0]$. Since $\rho(u)$ is unipotent, there is a nbhd $\mathcal{O}$ of $[v_0]$ which consists of points which are NOT $\rho(u)$-recurrent. On the other hand, let μ_* be the pushfoward of the G-invariant, finite, regular measure μ on G/Γ to G/H. By Poincare recurrence theorem a.e. point is $\rho(u)$-recurrent, which implies $\mu_*(\mathcal{O}) = 0$. It contradicts the regularity of μ_*. ■