Lecture 4: Selberg's local rigidity

Monday, January 16, 2017 10:56 AM

Our main goal today is to prove Selberg's local rigidity.

<u>Theorem (Selberg)</u> A cocompact lattice of $SL_n(\mathbb{R})$ has local rigidity for $n \geq 3$.

Let Γ be a cocompact lattice of $SL_n(\mathbb{R})$ and $\rho_t : \Gamma \to SL_n(\mathbb{R})$ be a deformation of the natural embedding $\rho_0 : \Gamma \to SL_n(\mathbb{R})$.

We'd like to prove $\exists g_t \in GL_n(\mathbb{R})$, such that $\rho_t(\gamma) = g_t \gamma g_t^{-1}$.

Here is the outline of proof:

<u>Step 1</u>. If $tr(\rho_t(\gamma)) = tr(\gamma)$ $\forall \gamma \in \Gamma$, then ρ_t is a trivial deformation. (It is enough to prove <u>trace rigidity</u>.)

<u>Step 2</u>. Let $\Gamma^{(r)} := \{ \gamma \in \Gamma \mid$ all of eigenvalues $\lambda_1, \dots, \lambda_n$ of γ are real and $|\lambda_i| \neq |\lambda_j|$ if $i \neq j \}$.

If $tr(\rho_t(\gamma)) = tr(\gamma)$ $\forall \gamma \in \Gamma^{(r)}$, then ρ_t is a trivial deformation. (It is enough to prove <u>trace rigidity</u> for $\mathbb{R}$-regular elements.)

<u>Step 3.</u> If $\rho_t(\Gamma)$ is a cocompact lattice and $\ker(\rho_t) = \{1\}$, then, for any $\gamma \in \Gamma^{(r)}$, $\rho_t(\gamma) \in \Gamma^{(r)}$ and

$$\log |\lambda_{i+1}(\rho_t(\gamma))| \,/\, \log |\lambda_i(\rho_t(\gamma))| \text{ is independent of } t.$$

(Makes use of Weyl chambers in a single flat.)

Lecture 4: Trace rigidity implies rigidity

Wednesday, January 18, 2017 11:20 AM

Step 4. If $\rho_t(\Gamma)$ is a cocompact lattice and $\ker(\rho_t)=1$, then for any $\gamma\in\Gamma^{(m)}$, $\lambda_i(\rho_t(\gamma))$ is independent of t.

(Makes use of two neighboring chambers in Tits's geometry.)

In particular, $tr(\rho_t(\gamma)) = tr(\gamma)$ for any $\gamma\in\Gamma^{(m)}$.

Step 5. If ρ_t is a deformation of a cocompact lattice Γ in $SL_n(\mathbb{R})$, then $\rho_t(\Gamma)$ is a cocompact lattice and $\ker(\rho_t)=1$.

Step 1. Borel's density theorem makes this the easiest step.

Lemma. In the above setting, it is enough to prove

$$tr(\gamma) = tr(\rho_t(\gamma)).$$

Proof. Let A_Γ be the $\mathbb{R}$-span of Γ in $M_n(\mathbb{R})$.

Claim ① $A_\Gamma = M_n(\mathbb{R})$.

Pf. If $A_\Gamma \neq M_n(\mathbb{R})$, then $\exists$ a linear functional $0\neq \ell: M_n(\mathbb{R})\to\mathbb{R}$ such that $\ell(A_\Gamma)=0$. So $\forall\gamma\in\Gamma$, $\ell(\gamma)=0$.

By Borel's denity theorem, Γ is Zarisiki-dense in $SL_n(\mathbb{R})$.

Hence $\ell(SL_n(\mathbb{R}))=0$. $\forall X\in M_n(\mathbb{R})$, $\exists\lambda\in\mathbb{R}$ s.t. $\det(X+\lambda I)>0$. Thus $\exists\,\lambda_1,\lambda_2\in\mathbb{R}$, $\lambda_1(X+\lambda_2 I)\in SL_n(\mathbb{R})$

$\Rightarrow$ The $\mathbb{R}$-span of $SL_n(\mathbb{R})$ is $M_n(\mathbb{R})$, which contradict $\ell\neq 0$ ∎

Lecture 4: Trace rigidity implies rigidity

Thursday, January 19, 2017 6:40 PM

①' (In class we used step 5 to say $\rho_t(\Gamma)$ is a lattice to deduce that the $\mathbb{R}$-span $\rho_t(\Gamma)$ is $M_n(\mathbb{R})$. Here is an easy why to get this for small t:

Let $\gamma_1, \ldots, \gamma_{n^2} \in \Gamma$ be an $\mathbb{R}$-basis of $M_n(\mathbb{R})$, and let $[\gamma_i]_{\mathcal{B}}$ be the vector of γ_i written in the standard basis $\{E_{ij} \mid 1 \leq i, j \leq n\}$ of $M_n(\mathbb{R})$. Then

$$\det \begin{bmatrix} [\gamma_1]_{\mathcal{B}} \\ \vdots \\ [\gamma_{n^2}]_{\mathcal{B}} \end{bmatrix} \neq 0.$$

So by continuity of ρ_t on t we get

$$\det \begin{bmatrix} [\rho_t(\gamma_1)]_{\mathcal{B}} \\ \vdots \\ [\rho_t(\gamma_{n^2})]_{\mathcal{B}} \end{bmatrix} \neq 0.$$

Hence $A_{\rho_t(\Gamma)} = M_n(\mathbb{R})$.)

Lecture 4: Trace rigidity implies rigidity

Monday, January 16, 2017 11:15 AM

(small t)

<u>Claim</u> 2 For $g_1, g_2 \in M_n(\mathbb{R})$, if, $\forall \gamma \in \Gamma$, $tr(g_1 \rho_t(\gamma)) = tr(g_2 \rho_t(\gamma))$

then $g_1 = g_2$.

<u>Proof.</u> $\forall \gamma \in \Gamma$, $tr((g_1 - g_2)\rho_t(\gamma)) = 0 \Rightarrow tr((g_1 - g_2) A_{\rho_t(\Gamma)}) = 0$

where $A_{\rho_t(\Gamma)}$ is the $\mathbb{R}$-span of $\rho_t(\Gamma) \Rightarrow \forall\ 1 \le i, j \le n$,

$$tr\left((g_1 - g_2) \begin{array}{cc} \begin{array}{c} j \end{array} & \\ \begin{bmatrix} 0 & \vdots & 0 \\ 0 \cdots 0 & 1 & 0 \cdots 0 \\ 0 & \vdots & 0 \end{bmatrix} & i \end{array}\right) = 0 \Rightarrow (g_1 - g_2)_{ij} = 0 \Rightarrow g_1 = g_2. \quad \square$$

<u>Claim</u> 3 If, $\forall \gamma \in \Gamma$, $tr(\gamma) = tr(\rho_t(\gamma))$, then

$$T_{\rho_t} : M_n(\mathbb{R}) \to M_n(\mathbb{R}),\quad T_{\rho_t}\left(\sum_{i=1}^{m} r_i \gamma_i\right) := \sum_{i=1}^{m} r_i\, \rho_t(\gamma_i)$$

is well-defined where $r_i \in \mathbb{R}$, $\gamma_i \in \Gamma$.

<u>Proof.</u> $\displaystyle\sum_{i=1}^{m} r_i^{(1)} \gamma_i^{(1)} = \sum_{i=1}^{m} r_i^{(2)} \gamma_i^{(2)}$

$$\Rightarrow \forall \gamma \in \Gamma,\ tr\left(\sum r_i^{(1)} \gamma_i^{(1)} \gamma\right) = tr\left(\sum r_i^{(2)} \gamma_i^{(2)} \gamma\right)$$

$$\Rightarrow \forall \gamma \in \Gamma,\ \sum r_i^{(1)}\, tr(\gamma_i^{(1)} \gamma) = \sum r_i^{(2)}\, tr(\gamma_i^{(2)} \gamma)$$

$$\Rightarrow \forall \gamma \in \Gamma,\ \sum r_i^{(1)}\, tr(\rho_t(\gamma_i^{(1)} \gamma)) = \sum r_i^{(2)}\, tr(\rho_t(\gamma_i^{(2)} \gamma))$$

$$\Rightarrow \forall \gamma \in \Gamma,\ tr\left(\underbrace{\left(\sum r_i^{(1)} \rho_t(\gamma_i^{(1)})\right)}_{g_1} \rho_t(\gamma)\right) = tr\left(\underbrace{\left(\sum r_i^{(2)} \rho_t(\gamma_i^{(2)})\right)}_{g_2} \rho_t(\gamma)\right)$$

$\Rightarrow$ using the previous <u>claim</u>, we get $\sum r_i^{(1)} \rho_t(\gamma_i^{(1)}) = \sum r_i^{(2)} \rho_t(\gamma_i^{(2)})$, which implies that T_{ρ_t} is well-defined. $\square$

Lecture 4: Trace rigidity implies rigidity

Monday, January 16, 2017 11:31 AM

Claim 4 $\exists g_t \in GL_n(\mathbb{R})$, $T_{\rho_t}(x) = g_t x g_t^{-1}$ for any $x \in M_n(\mathbb{R})$.

Pf. The way T_{ρ_t} is defined, we have that it is $\mathbb{R}$-linear.

$\forall x_1, x_2 \in M_n(\mathbb{R})$, $\exists r_i^{(1)}, r_i^{(2)} \in \mathbb{R}$ and $\gamma_i^{(1)}, \gamma_i^{(2)} \in \Gamma$ such that

$x_j = \sum r_i^{(j)} \gamma_i^{(j)}$. So

$$T_{\rho_t}(x_1 x_2) = T_{\rho_t}\left(\left(\sum_i r_i^{(1)} \gamma_i^{(1)}\right)\left(\sum_i r_i^{(2)} \gamma_i^{(2)}\right)\right)$$

$$= T_{\rho_t}\left(\sum_{i_1, i_2} r_{i_1}^{(1)} r_{i_2}^{(2)} \gamma_{i_1}^{(1)} \gamma_{i_2}^{(2)}\right)$$

$$= \sum_{i_1, i_2} r_{i_1}^{(2)} r_{i_2}^{(2)} \rho_t(\gamma_{i_1}^{(1)} \gamma_{i_2}^{(2)})$$

$$= \left(\sum_i r_i^{(1)} \rho_t(\gamma_i^{(1)})\right)\left(\sum_i r_i^{(2)} \rho_t(\gamma_i^{(2)})\right)$$

$$= T_{\rho_t}(x_1)\, T_{\rho_t}(x_2),$$

which implies $T_{\rho_t} : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ is an $\mathbb{R}$-algebra homomorphism. Hence $\exists g_t \in GL_n(\mathbb{R})$, $T_{\rho_t}(x) = g_t x g_t^{-1}$ for any $x \in M_n(\mathbb{R})$. (why ?). □

Hence $\forall \gamma \in \Gamma$, $\rho_t(\gamma) = T_{\rho_t}(\gamma) = g_t \gamma g_t^{-1}$. ■

<u>Remark</u> (Skolem-Noether) If A is a k-central simple algebra, then any k-automorphism $f: A \to A$ is inner, i.e. $\exists\, a \in U(A)$ (units), $\forall x \in A$, $f(x) = a x a^{-1}$.

Lecture 4: $\mathbb{R}$-regular elements

Monday, January 16, 2017 11:52 AM

Step 2 relies on our better understanding of $\mathbb{R}$-regular elements of $SL_n(\mathbb{R})$. Later we will their connection with <u>flats</u> in the symm. space.

<u>Definition</u> Let $G \subseteq GL_n(\mathbb{C})$ be Zariski-closed defined over $\mathbb{R}$. We say $g \in G(\mathbb{R})$ is $\mathbb{R}$-regular in G if $\exists g' \in GL_n(\mathbb{R})$ s.t.

$$g' C_G(g) g'^{-1} \subseteq \text{diagonal matrices},$$

where $C_G(g) := \{x \in G \mid gx = xg\}$ is the centralizer of g in $G(\mathbb{R})$

<u>Lemma</u>. $g \in SL_n(\mathbb{R})$ is $\mathbb{R}$-regular in $SL_n(\mathbb{R})$ if and only if eigenvalues of g are distinct real numbers.

<u>Proof</u>. ($\Rightarrow$) $\exists g' \in GL_n(\mathbb{R})$, $g' C_{SL_n(\mathbb{R})}(g) g'^{-1} \subseteq$ diag.

$\Rightarrow$ $g' g g'^{-1} \subseteq$ diag. matrices in $M_n(\mathbb{R})$ $\Rightarrow$ all eigenvalues of g are real numbers. So g is similar to a matrix of the form $\begin{bmatrix} \lambda_1 I_{n_1} & & \\ & \ddots & \\ & & \lambda_m I_{n_m} \end{bmatrix}$. The centralizer of $\begin{bmatrix} \lambda_1 I_{n_1} & & \\ & \ddots & \\ & & \lambda_m I_{n_m} \end{bmatrix}$ ($\lambda_i \neq \lambda_j$) in $M_n(\mathbb{R})$ is $\begin{bmatrix} M_{n_1}(\mathbb{R}) & & \\ & \ddots & \\ & & M_{n_m}(\mathbb{R}) \end{bmatrix}$. So $C_{SL_n(\mathbb{R})}(g)$ is a conjugate of

$$\left\{ d \in \begin{bmatrix} GL_{n_1}(\mathbb{R}) & & \\ & \ddots & \\ & & GL_{n_m}(\mathbb{R}) \end{bmatrix} \;\middle|\; \det d = 1 \right\}$$

which is commutative if and only if $n_1 = \cdots = n_m = 1$. This implies eigenvalues of g are distinct.

($\Leftarrow$) Since all the eigenvalues of g are distinct, the Jordan form of g is a diagonal matrix $\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}$. Since all the eigenvalues of g are real, $\exists\, g' \in GL_n(\mathbb{R})$ such that

$$g' g g'^{-1} = \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}, \text{ and } \lambda_i \neq \lambda_j.$$

So $C_{SL_n(\mathbb{R})}(g' g g'^{-1}) =$ diag. matrices in $SL_n(\mathbb{R})$

$\Rightarrow \quad g' C_{SL_n(\mathbb{R})}(g)\, g'^{-1} \subseteq$ diag. matrices. $\blacksquare$

<u>Remark</u>. Centralizer of a diagonal matrix can be understood using the following equation:

$$\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}\begin{bmatrix}x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn}\end{bmatrix}\begin{bmatrix}\lambda_1^{-1} & & \\ & \ddots & \\ & & \lambda_n^{-1}\end{bmatrix} = \begin{bmatrix}\lambda_i \lambda_j^{-1} x_{ij}\end{bmatrix}.$$

. The following property of $\mathbb{R}$-regular elements is crucial in the proof of the 2^{nd} step.

<u>Proposition</u>. Let $A := \left\{ \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix} \,\middle|\, \lambda_i \in \mathbb{R}^+ \right\}$ and, for $c \geq 1$, let $A_c := \left\{ \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix} \,\middle|\, \lambda_i/\lambda_{i+1} \geq c \right\}$. Then, for any nbhd $\mathcal{O}_G$ of I in $SL_n(\mathbb{R})$, any nbhd $\mathcal{O}_A$ of I in A, and any $c > 1$, there is a nbhd U_G of I in $SL_n(\mathbb{R})$ such that, $\forall a \in A_c$,

$$U_G\, a\, U_G \subseteq \{ g a d g^{-1} \mid g \in \mathcal{O}_G,\ d \in \mathcal{O}_A \}.$$

Lecture 4: Stability of $\mathbb{R}$-regularity

Tuesday, January 17, 2017 3:59 PM

<u>Corollary</u>. If g is $\mathbb{R}$-regular with positive eigenvalues, and $\eta>1$, then there is a nbhd U_G of I in G such that for any $i\in\mathbb{Z}^+$ and any $x\in U_G\, g^i\, U_G$ the eigenvalues $\lambda_j(x)$ of x satisfy:

① $\lambda_1(x)>\lambda_2(x)>\cdots>\lambda_n(x)>0$

② $\frac{1}{\eta}\lambda_j(g)^i<\lambda_j(x)<\eta\,\lambda_j(g)^i$ where $\lambda_1(g)>\cdots>\lambda_n(g)>0$ are eigen-values of g.

<u>Proof</u>. Since g is $\mathbb{R}$-regular, $\exists\, g'\in GL_n(\mathbb{R})$ such that

$g'g\,g'^{-1}=\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}$ and $\lambda_i\neq\lambda_j$. And by our assumption $\lambda_i>0$. After conjugating by a permutation matrix, if needed, we can and will assume $\lambda_1>\lambda_2>\cdots>\lambda_n$.

Hence w.l.o.g. we can and will assume $g\in A_c$ for some $c>1$. So, for any $i\in\mathbb{Z}^+$, $g^i\in A_c$.

Now let $O_A:=\{\operatorname{diag}(\alpha_1,\ldots,\alpha_n)\mid \frac{1}{\sqrt[4]{c}}<\alpha_i<\sqrt[4]{c}$ $\frac{1}{\eta}<\alpha_i<\eta\}$. Then for any $a\in A_c$ and $d\in O_A$, we have

$$ad=\operatorname{diag}(a_1,\ldots,a_n)\cdot\operatorname{diag}(d_1,\ldots,d_n)=\operatorname{diag}(a_1d_1,\ldots,a_nd_n)$$

and $$a_id_i/_{a_{i+1}d_{i+1}}=\frac{a_i}{a_{i+1}}\cdot d_i\cdot d_{i+1}^{-1}\geq c\cdot(\sqrt[4]{c})^{-1}\cdot(\sqrt[4]{c})^{-1}=\sqrt{c}>1$$

So $ad\in A_{\sqrt{c}}$ is $\mathbb{R}$-regular. Moreover $\frac{1}{\eta}a_i<a_id_i<\eta\, a_i$.

Lecture 4: Stability of $\mathbb{R}$-regularity

Tuesday, January 17, 2017 10:35 AM

Let $\mathcal{O}_G$ be any nbhd of I in $SL_n(\mathbb{R})$, and let U_G be the nbhd of I which Proposition gives us for the parameters $\mathcal{O}_G, \mathcal{O}_A$, and c as above. Then, for any $i \in \mathbb{Z}^+$, $U_G g^i U_G \subseteq \{x g^i d x^{-1} \mid x \in \mathcal{O}_G, d \in \mathcal{O}_A\}$.

So by the above discussion:

1. $g^i d$ is $\mathbb{R}$-regular with positive eigen-values.
2. $\frac{1}{\eta} \lambda_j(g)^i < \lambda_j(g^i d) < \eta\, \lambda_j(g)^i$.

And so for any $g' \in SL_n(\mathbb{R})$ we get

1. $g' g^i d g'^{-1}$ is $\mathbb{R}$-regular with positive eigen-values
2. $\frac{1}{\eta} \lambda_j(g)^i < \lambda_j(g' g^i d g'^{-1}) < \eta\, \lambda_j(g)^i$

Hence for any $x \in U_G g^i U_G$ we have

1. x is $\mathbb{R}$-regular with positive eigen-values.
2. $\frac{1}{\eta} \lambda_j(g)^i < \lambda_j(x) < \eta\, \lambda_j(g)^i$. ■

Remark. The importance of the above corollary is on the fact that a single nbhd U_G of I can work for all of positive powers of g at the same time. It is much easier to show that the set of $\mathbb{R}$-regular elements of $SL_n(\mathbb{R})$ is open, but that gives us a nbhd which works only for a single $i \in \mathbb{Z}^+$.