Lecture 8: Step 3 of Selberg's proof of local rigidity

Sunday, January 22, 2017 6:18 PM

$\boxed{\text{Step 3}}$ Let $\gamma \in \Gamma^{(r)}$.

① Then $\Gamma_\gamma := C_{SL_n(\mathbb{R})}(\Gamma)$ is abelian.

② Let $\Gamma_\gamma^+ := \{\gamma' \in \Gamma_\gamma \mid \gamma' \text{ has positive eigen-values}\}$. Then

$\Gamma_\gamma^+ \simeq \mathbb{Z}^{n-1}$.

③ Suppose $\rho_t(\Gamma)$ is a cocompact lattice in $SL_n(\mathbb{R})$ and $\ker \rho_t = 1$.

Then
$$\frac{\log \lambda_i(\rho_t(a))}{\log \lambda_j(\rho_t(a))} = \frac{\log \lambda_i(a)}{\log \lambda_j(a)} \quad \text{for any } a \in \Gamma_\gamma^+$$

where as always $\lambda_i(\cdot)$ is the i^{th} eigenvalue of a in some ordering. Similarly

$$\frac{\log |\lambda_i(\rho_t(a'))|}{\log |\lambda_j(\rho_t(a'))|} = \frac{\log |\lambda_i(a')|}{\log |\lambda_j(a')|} \quad \text{for any } a' \in \Gamma_\gamma.$$

Lemma. Suppose Γ is a cocompact lattice in G. Then, for any $\gamma \in \Gamma$, $C_\Gamma(\gamma)$ is a cocompact lattice in $C_G(\gamma)$.

Proof. There is a natural embedding $C_G(\gamma)/C_\Gamma(\gamma) \hookrightarrow G/\Gamma$.

It is enough to show it is a proper map, i.e. preimage of a compact set is compact.

Lecture 8: Centralizer of an element of Γ

Monday, January 23, 2017 3:16 AM

Let C be a compact subset of G. The preimage of $C\Gamma/\Gamma$ under the above embedding is $\left(C\Gamma \cap C_G(\gamma)\right) C_\Gamma(\gamma)/C_\Gamma(\gamma)$.

Suppose $(c\,\lambda)\gamma = \gamma\,(c\,\lambda)$. So

$$\lambda\gamma\lambda^{-1} = c^{-1}\gamma c \in \underbrace{C^{-1}\gamma C}_{\text{compact}} \cap \underbrace{\Gamma}_{\text{discrete}}.$$

Therefore there is a finite set $\tilde{F}$ of conjugates of γ which can be written as $\lambda\gamma\lambda^{-1}$ where $\lambda \in \Gamma$ and $\exists c \in C$ s.t. $c\lambda \in C_G(\gamma)$. Hence

$$\{\lambda C_\Gamma(\gamma) \mid \exists c \in C, c\lambda \in C_G(\gamma)\} =: F$$

is a finite subset of $G/C_\Gamma(\gamma)$.

So $\left(C\Gamma \cap C_G(\gamma)\right) C_\Gamma(\gamma)/C_\Gamma(\gamma) \subseteq \underbrace{CF/C_\Gamma(\gamma)}_{\text{compact}}$. $\blacksquare$

$\boxed{\text{Some of its consequences}}$

If $\gamma \in \Gamma^{(r)}$, then $\exists g \in SL_n(\mathbb{R})$, $g\, C_{SL_n(\mathbb{R})}(\gamma)\, g^{-1} \subseteq \text{diag}$.

$\Rightarrow C_\Gamma(\gamma) =: \Gamma_\gamma$ is abelian; and $g\Gamma_\gamma^+ g^{-1}$ is a cocompact lattice in $\left\{ \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} \mid \prod a_i = 1, a_i \in \mathbb{R}^+ \right\} \xrightarrow{\sim} \{(x_1, \dots, x_n) \in \mathbb{R} \mid \sum x_i = 0\}$

Lecture 8: Proof of Step 3: cocompact flats

Monday, January 23, 2017 3:37 AM

Hence $C_\Gamma(\gamma)$ is isomorphic to a lattice in $\mathbb{R}^{n-1}$. Hence

$$C_\Gamma(\gamma) \simeq \mathbb{Z}^{n-1} \quad \text{(why ?)},$$

which implies parts ① and ② of Step 3.

<u>Claim</u>. If Γ is a cocompact lattice in $SL_n(\mathbb{R})$, then any element of Γ is semisimple, i.e. diagonalizable over $\mathbb{C}$.

<u>Proof of claim.</u> Any $\gamma \in \Gamma$ can be written in its Jordan form over $\mathbb{C}$, which implies that $\gamma = s.u$ where s is semisimple, u is unipotent, and $su = us$. So $u \in C_{SL_n(\mathbb{R})}(s)$.

Using "the real version" of Jordan form, one can see that $\exists\, a_i \in C_{SL_n(\mathbb{R})}(s)$ s.t. $a_i u a_i^{-1} \longrightarrow I$. So $a_i \gamma a_i^{-1} \xrightarrow[i\to\infty]{} s$.

Hence $s \in$ Closure of the conjugacy class of γ. On the other hand, let $\mathcal{F}$ be a compact set s.t. $\mathcal{F}\Gamma = G$, then $\{g\gamma g^{-1} \mid g \in G\} = \{k\gamma'\gamma\gamma'^{-1}k^{-1} \mid k \in \mathcal{F}, \gamma' \in \Gamma\}$ which is closed. So s is a conjugate of γ. ∎

<u>Claim</u>. Suppose $\gamma \in \Gamma^{(r)}$, $\rho_t(\Gamma)$ is a cocompact lattice in $SL_n(\mathbb{R})$, and $\ker(\rho_t) = I$. Then $\rho_t(\gamma)$ is $\mathbb{R}$-regular.

Lecture 8: Proof of Step 3: cocompact flats

Monday, January 23, 2017 9:17 AM

<u>Proof of Claim</u>. By the above discussion, $C_{\Gamma}(\gamma)$ is abelian. Hence $C_{\rho_t(\Gamma)}(\rho_t(\gamma))$ is abelian, and at the same time a cocompact lattice in $C_{SL_n(\mathbb{R})}(\rho_t(\gamma))$. Now suppose t_0 is the smallest value at which $\rho_{t_0}(\gamma)$ has two equal eigen-values. In particular, for any $0 \le t \le t_0$, all of eigen-values of $\rho_t(\gamma)$ are real. By the previous claim, $\rho_{t_0}(\gamma)$ is diago. over $\mathbb{R}$ Hence $C_{SL_n(\mathbb{R})}(\rho_{t_0}(\gamma))$ contains a copy of $SL_2(\mathbb{R})$. So the following version of Borel's density theorem, implies a copy of $SL_2(\mathbb{R})$ is in the Zariski-closure of $C_{\rho_{t_0}(\Gamma)}(\rho_{t_0}(\gamma))$, which implies $C_{\rho_0(\Gamma)}(\rho_{t_0}(\gamma)) = \rho_{t_0}(C_{\Gamma}(\gamma))$ is NOT abelian. That is a contradiction. $\blacksquare$

Theorem (Borel's density theorem) Suppose Γ is a lattice in $G(\mathbb{R})$ where G is an alg. group$/\mathbb{R}$. Then the Zariski-closure H of Γ contains $G(\mathbb{R})^+ := \langle u \mid u \in G(\mathbb{R}), \text{unip.} \rangle$.

<u>Pf.</u> By Chevalley's theorem, $\exists \rho: G \to GL(V)$ and $[v_0] \in \mathbb{P}(V)$ s.t. $H = \{g \in G \mid \rho(g)[v_0] = [v_0]\}$. If $G(\mathbb{R})^+ \not\subseteq H$, then there is a unip. $u \in G(\mathbb{R})$ s.t. $\rho(u)[v_0] \neq [v_0]$.

Lecture 8: Proof of Step 3: chamber preserving maps

Monday, January 23, 2017 2:30 PM

$\exists$ a nbhd $\mathcal{O}$ of $[v_0]$ s.t. $\left|\{n\in\mathbb{Z}^+ \mid \rho(u^n)[v]\in\mathcal{O}\}\right|<\infty$,

for any $[v]\in\mathcal{O}$, i.e. no point in $\mathcal{O}$ is a $\rho(u)$-recurrent point which contradicts Poincaré's recurrence theorem. $\blacksquare$

Suppose $\gamma\in\Gamma^{(r)}$. After changing ρ_t by a continuous conjugation and Γ by a conjugation, we can assume

$$C_\Gamma(\gamma)\subseteq \text{diag}. \quad \text{and} \quad C_{\rho_t(\Gamma)}(\rho_t(\gamma))\subseteq \text{diag}.$$

Hence $\Delta_t := \{\log x \mid x=y^2 \text{ for some } y\in C_{\rho_t(\Gamma)}(\rho_t(\gamma))\}$ is a lattice in $\mathfrak{a}:=\{(x_1,\dots,x_n)\in\mathbb{R}^n \mid \sum_{i=1}^n x_i=0\}$. For a permutation $\sigma\in S_n$, let $\mathfrak{a}_\sigma^+ := \{\vec{x}\in\mathfrak{a} \mid x_{\sigma_1}<x_{\sigma_2}<\cdots<x_{\sigma_n}\}$. Then $\exists$ isomorphisms $\Theta_t:\Delta_0\to\Delta_t$ s.t. ① Θ_t is continuous w.r.t. t

② $\Theta_t\left(\bigcup_{\sigma\in S_n}(\Delta_0\cap\mathfrak{a}_\sigma^+)\right)=\bigcup_{\sigma\in S_n}(\Delta_t\cap\mathfrak{a}_\sigma^+)$.

Since $\mathfrak{a}_\sigma^+$'s are connected components of $\bigcup_{\sigma\in S_n}\mathfrak{a}_\sigma^+$ and, for any $v\in\Delta_0\cap\mathfrak{a}_\sigma^+$, $v=\Theta_0(v)$ is path-connected to $\Theta_t(v)$, we have

$$\forall\sigma,\ \Theta_t(\Delta_0\cap\mathfrak{a}_\sigma^+)=\Delta_t\cap\mathfrak{a}_\sigma^+.$$

Lecture 8: Proof of Step 3: chamber preserving maps

Monday, January 23, 2017 3:09 PM

So to finish Proof of Step 3, it is enough to prove the following lemma:

Lemma. Let Δ, Δ' be two lattices in $\mathfrak{a} := \{\vec{x} \in \mathbb{R}^n \mid \sum x_i = 0\}$.

Let $\theta : \mathfrak{a} \to \mathfrak{a}$ be a linear bijection. Suppose

① $\theta(\Delta) = \Delta'$ ② $\theta(\Delta \cap \mathfrak{a}_\sigma^+) = \Delta' \cap \mathfrak{a}_\sigma^+$ for any $\sigma \in S_n$.

Then θ is scaling, i.e. $\exists c \in \mathbb{R}^+$, $\theta(v) = cv$

We start with the following Lemma.

(directional density of lattices)

Lemma. Let V be a finite-dimensional $\mathbb{R}$-vector space, and let Δ be a lattice in V. Then $\{\mathbb{R}v \mid v \in \Delta \setminus \{0\}\}$ is dense in the projective space $\mathbb{P}(V)$.

proof. Since Δ is a lattice in V, $\exists\, v_1, \dots, v_n \in \Delta$ s.t.

$\Delta = \bigoplus_{i=1}^n \mathbb{Z} v_i$ and $V = \bigoplus_{i=1}^n \mathbb{R} v_i$. As $(c_1, \dots, c_n) \mapsto \sum c_i v_i$

Lecture 8: Proof of Step 3: chamber preserving maps

Tuesday, January 24, 2017 8:39 AM

induces a homeomorphism from $P(\mathbb{R}^n)$ to $P(V)$, we can and will assume that $\Delta = \mathbb{Z}^n$ and $V = \mathbb{R}^n$. Now we deduce the claim using the facts that $\{\mathbb{R}v \mid v \in \mathbb{Z}^n \setminus 0\} = \{\mathbb{R}v \mid v \in \mathbb{Q}^n \setminus 0\}$ and $\mathbb{Q}$ is dense in $\mathbb{R}$. $\blacksquare$

<u>Corollary</u>. Let V be a subspace, let Δ_1, Δ_2 be lattices in V, let C be a cone base at the origin, i.e. $v \in C \Rightarrow \mathbb{R}^+ v \in C$. Suppose $\overline{C^\circ} = C$. Let $\theta: V \to V$ be an $\mathbb{R}$-linear isomorphism s.t.

① $\theta(\Delta_1) = \Delta_2$ ② $\theta(\Delta_1 \cap C) = \Delta_2 \cap C$.

Then $\theta(C) = C$.

<u>Proof</u>. $\forall\ v \in C$, by the density of directions in Δ_1, $\exists\ v_{n,\Delta_1} \in \Delta_1$ s.t. $\mathbb{R}v_{n,\Delta} \longrightarrow \mathbb{R}v$. Changing v_{n,Δ_1} to $-v_{n,\Delta_1}$ if needed, we can assume that $\mathbb{R}^+ v_{n,\Delta_1} \longrightarrow \mathbb{R}^+ v$. For $n \gg 1$, we have $v_{n,\Delta_1} \in \Delta_1 \cap C$. Hence $\theta(v_{n,\Delta_1}) \in \Delta_1 \cap C$. Therefore

Lecture 8: Proof of Step 3: chamber preserving maps

Tuesday, January 24, 2017 9:01 AM

$\dfrac{\theta(v_{n,\Delta_1})}{\|\theta(v_{n,\Delta_1})\|} \in \{u \in C \mid \|u\|=1\} =: C^1$. Hence $\dfrac{\theta(v)}{\|\theta(v)\|} \in \overline{C^1}$.

Since θ is an isomorphism, $w \overset{\theta_1}{\longmapsto} \dfrac{\theta(w)}{\|\theta(w)\|}$ is a homeomorphism from $\{u \in V \mid \|u\|=1\}$ to itself. Hence $\theta_1(C^1)$ is open. Therefore by the above argument $\theta_1(C^1) \subseteq \overline{C^1}^{\circ}$. Since $\overline{C}^{\circ} = C$, we get $\theta_1(C^1) \subseteq C^1$. Similarly we get $\theta_1^{-1}(C^1) \subseteq C^1$. So $\theta_1(C^1) = C^1$, and $\theta(C) = C$. ■

<u>Let's go back to the proof of Lemma.</u>

Using the above corollary, for any $\sigma \in S_n$, $\theta(\mathfrak{X}_\sigma^+) = \mathfrak{X}_\sigma^+$.

Notice that $\mathfrak{X}_\sigma^+$ has

$n-1$ walls (codim. 1), $\binom{n-1}{2}$ faces of codim 2, $\dots$, $\binom{n-1}{k}$ faces of codim k, $\dots$, $n-1$ faces of dim 1.

Any face of codim $k+1$ is intersection of exactly two faces of codim k. Hence by induction we have that $\theta(\mathcal{F}) = \mathcal{F}$ for any face $\mathcal{F}$ of $\mathfrak{X}_\sigma^+$. In particular one-dimensional faces are eigen-directions of θ. One of 1-dimensional faces

Lecture 8: Proof of Step 3: chamber preserving maps

Tuesday, January 24, 2017 9:09 AM

of $\mathfrak{X}^+_{id}$ is $\mathbb{R}^+(n-1,-1,\dots,-1)$ which is $\{(x_1,\dots,x_n) \mid x_1 > x_2 = \dots = x_n, \sum x_i = 0\}$.

Hence $\mathbb{R}^+(-1,\dots,n-1,\dots,-1)$ is a 1-dim face of $\mathfrak{X}^+_{(1,n-1)}$.

Therefore $\exists c_i \in \mathbb{R}^+$ s.t. $\theta(ne_i - \sum_{j=1}^n e_j) = c_i (ne_i - \sum_{j=1}^n e_j)$.

Let $v_k = ne_k - \sum_{j=1}^n e_j$. So $\sum_{i=1}^n v_i = 0$ and $v_1,\dots,v_{n-1}$ are linearly independent. Thus

$$-v_n = \sum_{i=1}^{n-1} v_i \implies \theta(-v_n) = \sum_{i=1}^{n-1} \theta(v_i)$$

$$\implies -c_n v_n = \sum_{i=1}^{n-1} c_i v_i$$

$$\implies -v_n = \sum_{i=1}^{n-1} \frac{c_i}{c_n} v_i$$

Since $v_1,\dots,v_{n-1}$ are linearly independent $\frac{c_i}{c_n} = 1$ for any i. So $\theta(v_i) = c\, v_i$, which implies $\theta(v) = cv$ for any $v \in V$. ∎

<u>Remark</u>. Here we proved "irreducible spherical apartments are rigid under linear maps": The simplicial decomposition attached to $\{\mathfrak{X}^+_\sigma\}$ on the unit sphere.