Lecture 9: Proof of step 4: either fixing eigenvalues or having a rational lattice

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For $\gamma_0 \in \Gamma^{(r)}$, let $g_0 \in SL_n(\mathbb{R})$ be an element such that

$g_0 \gamma g_0^{-1} = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ where $\lambda_1 > \cdots > \lambda_n > 0$. Then as we said

$$\Delta_{\gamma_0} := g_0 \{ \operatorname{diag}(x_1, \dots, x_n) \mid x_i \in \mathbb{R}, \sum x_i = 0, g_0^{-1} \operatorname{diag}(e^{x_1}, \dots, e^{x_n}) g_0 \in \Gamma \} g_0^{-1}$$

is a lattice in $\operatorname{Lie} C_G(\gamma_0)^\circ = g_0 \{ \operatorname{diag}(x_1, \dots, x_n) \mid x_i \in \mathbb{R}, \sum x_i = 0 \} g_0^{-1}$.

Let $\Delta_{\mathbb{Q}}(\gamma_0) := g_0 \{ \operatorname{diag}(x_1, \dots, x_n) \mid x_i \in \mathbb{Z}, \sum x_i = 0 \} g_0^{-1}$.

So both Δ_{γ_0} and $\Delta_{\mathbb{Q}}(\gamma_0)$ are lattices in $\operatorname{Lie} C_G(\gamma_0)$.

So far we have proved, if ρ_t is a continuous deformation s.t. $\rho_t(\Gamma)$ is a cocompact lattice and $\ker \rho_t = I$, then ① $\rho_t(\Gamma^{(r)}) = \rho_t(\Gamma)^{(r)}$, and ② $\forall \gamma_0 \in \Gamma$, there are continuous functions $c(t) \in \mathbb{R}^+$, $g_t \in G$ s.t.

$$\rho_t(\gamma) = g_t \gamma^{c(t)} g_t^{-1} \quad \text{for any } \gamma \in \exp(\Delta_{\gamma_0}).$$

Next we want to prove that

<u>Lemma</u>. In addition to the above assumptions, let's assume $\{\rho_t\}$ is a regular deformation of ρ_0, i.e. $\forall \gamma \in \Gamma$, $t \mapsto \rho_t(\gamma)$ has entries in $\mathbb{R}[t]$. Then either ① $\rho_t(\gamma) = g_t \gamma g_t^{-1}$ for any $\gamma \in \exp(\Delta_{\gamma_0})$ or ② $c_0 \Delta_{\gamma_0} \cap \Delta_0(\gamma_0)$ is a finite-index subgroup of $\Delta_0(\gamma_0)$ for some c_0.

<u>Proof of lemma</u>. For a given $\gamma_0 \in \Gamma^{(r)}$, after a continuous conjugation,

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we can assume that $\rho_t(\gamma_0)$ is diag. for any t. So $\forall \gamma \in e^{\Delta_{\gamma_0}}$

$\rho_t(\gamma) = \mathrm{diag}(p_1(t), \dots, p_n(t)) = \mathrm{diag}(\lambda_1^{c(t)}, \dots, \lambda_n^{c(t)})$.

Hence $p_i(t) = \lambda_i^{c(t)} = e^{c(t)\ln\lambda_i} = \lambda_j^{c(t)\frac{\ln\lambda_i}{\ln\lambda_j}} = p_j(t)^{\frac{\ln\lambda_i}{\ln\lambda_j}}$ ⊛

Choosing a simply-connected region of $\mathbb{C}$ which contains the interval of t where ⊛ holds and avoids zeros of p_i and p_j, we can define analytic functions $\ln p_i(z)$ and $\ln p_j(z)$.

By ⊛ we get $\ln p_i(z) = \frac{\ln\lambda_i}{\ln\lambda_j}\ln p_j(z)$. We make sure that $[x_0,\infty)$ is in the considered simply-connected region. For large enough $t\in\mathbb{R}$, we get $p_i(t)^2 = p_j(t)^{2\frac{\ln\lambda_i}{\ln\lambda_j}}$.

Now, by growth rate comparison, we get

$$\deg p_i = \frac{\ln\lambda_i}{\ln\lambda_j}\deg p_j.$$

So either $\deg p_i = \deg p_j = 0$ <u>or</u> $\frac{\ln\lambda_i}{\ln\lambda_j}\in\mathbb{Q}$.

Hence <u>either</u> $\rho_t(\gamma)=\gamma$ for any t and any $\gamma\in e^{\Delta_{\gamma_0}}$,

<u>or</u> $\forall v\in\Delta_{\gamma_0}$, $\exists c_v\in\mathbb{R}^+$ s.t. $c_v v\in\Delta_0(\gamma_0)$.

Let $v_1,\dots,v_{n-1}$ be a basis of Δ_{γ_0}. Then $\exists c_1,\dots,c_{n-1}\in\mathbb{R}^+$

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s.t. $v_1' = c_1 v_1, \dots, v_{n-1}' = c_{n-1} v_{n-1} \in \{\mathrm{diag}(x_1, \dots, x_n) \mid x_i \in \mathbb{Q}, \sum x_i = 0\}$

form a $\mathbb{Q}$-basis. And there is $c_0 \in \mathbb{R}^+$ s.t.

$c_0(v_1 + \dots + v_{n-1}) \in \Delta_0(\gamma_0) \Rightarrow c_0(v_1 + \dots + v_{n-1})$ can be written

as a $\mathbb{Q}$-linear combination of $v_1', \dots, v_{n-1}'$. So

$\exists\, r_i \in \mathbb{Q}$ s.t. $r_1 v_1' + \dots + r_{n-1} v_{n-1}' = c_0 v_1 + \dots + c_0 v_{n-1}$

$$= c_0 c_1^{-1} v_1' + \dots + c_0 c_{n-1}^{-1} v_{n-1}.$$

$\Rightarrow c_0 c_i^{-1} = r_i \in \mathbb{Q} \Rightarrow c_0 \Delta_{\gamma_0} \subseteq \mathbb{Q}$-span of $\Delta_0(\gamma_0)$

$\Rightarrow c_0 \Delta_{\gamma_0} \cap \Delta_0(\gamma_0)$ is a finite-index subgroup of $\Delta_0(\gamma_0)$. ■

Now we show in the second case of the above lemma, again eigen-values should be preserved, which finishes proof of step 4.

As above assume, for any t, $\rho_t(\gamma_0)$ is diagonal; and assumed $\rho_t(\gamma_0) \neq \gamma_0$. Then ① $\forall\, \gamma \in e^{\Delta_{\gamma_0}}$, $\rho_t(\gamma) = \gamma^{c(t)}$

② $\exists\, c_0 \in \mathbb{R}^+$, $\mathbb{Q}$-span of $\Delta_{\gamma_0} =$

$$c_0 \{\mathrm{diag}(x_1, \dots, x_n) \mid x_i \in \mathbb{Q}, \textstyle\sum x_i = 0\}.$$

So $\exists\, \gamma_1 \in C_\Gamma(\gamma_0) \cap T^{(r)}$ s.t. $\gamma_1 = \mathrm{diag}(\lambda_1, \lambda_1, \lambda_2, \dots, \lambda_{n-1})$ and $\lambda_1 > \lambda_2 > \dots > \lambda_{n-1} > 0$.

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So $C_\Gamma(\gamma_1)$ is a cocompact lattice in

$$C_{SL_n(\mathbb{R})}(\gamma_1)=\left\{\begin{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{matrix} & 0 \\ 0 & \begin{matrix} a_3 & & \\ & \ddots & \\ & & a_n\end{matrix}\end{bmatrix}\in SL_n(\mathbb{R})\right\}.$$

Let H be the above group, and let $\Lambda=\Gamma\cap H$. Since, for any t,

$$\rho_t(\gamma_1)=\gamma_1^{c_\circ(t)},\quad \rho_t(\Lambda)=\rho_t(C_\Gamma(\gamma_1))=C_{\rho_t(\Gamma)}(\gamma_1^{c_\circ(t)})=\rho_t(\Gamma)\cap H.$$

- Let $\Theta: H^\circ\to SL_2(\mathbb{R})\times(\mathbb{R}^+)^{n-2}$,

$$\Theta\left(\begin{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{matrix} & \\ & \begin{matrix} a_3 & & \\ & \ddots & \\ & & a_n\end{matrix}\end{bmatrix}\right)=\left(\frac{1}{\sqrt{a_{11}a_{22}-a_{12}a_{21}}}\begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix},\ (a_3,\dots,a_n)\right).$$

- One can check that Θ is a (Lie group) isomorphism.

<u>Claim</u>. Suppose Λ is a cocompact lattice in H, and $\exists\,\gamma_\circ\in\Lambda\cap \mathrm{diag}$. which is $\mathbb{R}$-regular; Then $pr_{SL_2(\mathbb{R})}(\Theta(\Lambda))$ is a cocompact lattice in $SL_2(\mathbb{R})$.

<u>Proof of claim.</u> ① Suppose $\mathcal{F}$ is a compact subset of H s.t. $\mathcal{F}\Lambda=H$; then $\Psi(\mathcal{F})\Psi(\Lambda)=SL_2(\mathbb{R})$ where $\Psi=pr_{SL_2(\mathbb{R})}\circ\Theta$. So it is enough to prove $\Psi(\Lambda)$ is discrete. Suppose to the contrary that $\exists\,\lambda_i\in\Lambda$ s.t. $\Psi(\lambda_i)\neq I$ and $\Psi(\lambda_i)\to I$.

For any i, $\lambda_i=\begin{bmatrix}\Psi(\lambda_i) & \\ & I\end{bmatrix}d_i$ where

$$d_i=\mathrm{diag}(\sqrt{a_{11}a_{22}-a_{12}a_{21}},\sqrt{a_{11}a_{22}-a_{12}a_{21}},a_3,\dots,a_n).$$

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So $$\lambda_i \gamma_0 \lambda_i^{-1} = \begin{bmatrix} \Psi(\lambda_i) & \\ & I \end{bmatrix} d_i \gamma_0 d_i^{-1} \begin{bmatrix} \Psi(\lambda_i)^{-1} & \\ & I \end{bmatrix}$$

$$= \begin{bmatrix} \Psi(\lambda_i) & \\ & I \end{bmatrix} \gamma_0 \begin{bmatrix} \Psi(\lambda_i)^{-1} & \\ & I \end{bmatrix} \xrightarrow[i\to\infty]{} \gamma_0$$

Since Λ is discrete and $\lambda_i \gamma_0 \lambda_i^{-1} \in \Lambda$, we have $\lambda_i \gamma_0 \lambda_i^{-1} = \gamma_0$ for $i \gg 1$. Hence $\lambda_i \in C_H(\gamma_0) \Rightarrow \lambda_i$'s are diagonal.

Suppose $\lambda_i = \operatorname{diag}(\alpha_1^{(i)}, \ldots, \alpha_n^{(i)})$. Then, by assumption, $\frac{\alpha_1^{(i)}}{\alpha_2^{(i)}} \xrightarrow[i\to\infty]{} 1$.

Since Λ is a lattice in H, by the virtue of proof of Borel density, $\begin{bmatrix} SL_2(\mathbb{R}) & \\ & I \end{bmatrix}$ is a subgroup of the Zariski-closure of Λ.

In particular, $\exists \lambda = \begin{bmatrix} a_{11} & a_{12} & \\ a_{21} & a_{22} & \\ & & I \end{bmatrix} \in \Lambda$ s.t. $a_{12} a_{21} \neq 0$. So

$$\lambda_i \lambda \lambda_i^{-1} = \begin{bmatrix} a_{11} & \alpha_1^{(i)}/\alpha_2^{(i)}\, a_{12} & \\ \alpha_2^{(i)}/\alpha_1^{(i)}\, a_{21} & a_{22} & \\ & & I \end{bmatrix} \xrightarrow[i\to\infty]{} \lambda.$$ Again by discreteness of Λ, we have $\lambda_i \lambda \lambda_i^{-1} = \lambda$ if $i \gg 1$.

Therefore $\alpha_1^{(i)} = \alpha_2^{(i)}$ for $i \gg 1$, which implies $\Psi(\lambda_i) = 1$.

That is a contradiction. ■

<u>Corollary</u>. ρ_t induces a deformation $\rho_t' : \Psi(C_\Gamma(\gamma_1)) \longrightarrow \Psi(C_{\rho_t(\Gamma)}(\rho_t(\gamma_1)))$

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of cocompact lattices of $SL_2(\mathbb{R})$, where

$$\rho_t'(\Psi(\lambda)) := \Psi(\rho_t(\lambda)).$$

<u>Proof of corollary</u>. In our setting, $\rho_t(\gamma_0) = \gamma_0^{c_0(t)} \in \rho_t(C_{\rho_t(\Gamma')}(\rho_t(\gamma_0)))$ is diagonal and $\mathbb{R}$-regular. So, by the previous claim,

$\Lambda_t^{(2)} := \Psi(\rho_t(C_{\rho_t(\Gamma')}(\rho_t(\gamma_0))))$ is a cocompact lattice in $SL_2(\mathbb{R})$.

So it is enough to show ρ_t' is well-defined:

$$\Psi(\lambda_1) = \Psi(\lambda_2) \Leftrightarrow \lambda_2^{-1}\lambda_1 = \operatorname{diag}(a_1, a_1, a_3, \dots, a_n)$$

$$\Leftrightarrow \lambda_2^{-1}\lambda_1 \in C_H\left(\begin{bmatrix} SL_2(\mathbb{R}) & \\ & I \end{bmatrix}\right) \underset{\substack{\uparrow \\ \boxed{\text{By Borel's density argum.}}}}{=} C_H\left(\begin{bmatrix} SL_2(\mathbb{R}) & \\ & I \end{bmatrix} \cap \Gamma\right)$$

$$= C_H(H \cap \Gamma) = C_H(C_\Gamma(\gamma_1))$$

$$\Leftrightarrow \lambda_2^{-1}\lambda_1 \in C_{C_\Gamma(\gamma_1)}(C_\Gamma(\gamma_1))$$

$$\Leftrightarrow \rho_t(\lambda_2^{-1}\lambda_1) \in C_{\underbrace{C_{\rho_t(\Gamma')}(\rho_t(\gamma_1))}_{\boxed{\text{as we said, a lattice in } H}}}(C_{\rho_t(\Gamma')}(\rho_t(\gamma_1))) \subseteq C_H(C_{\rho_t(\Gamma')}(\gamma_1^{c_0(t)})) \underset{\substack{\uparrow \\ \boxed{\text{By Borel's density}}}}{=} Z(H)$$

$$\Leftrightarrow \Psi(\rho_t(\lambda_1)) = \Psi(\rho_t(\lambda_2)). \qquad \blacksquare$$