Lecture 11: Geometry of symmetry spaces

Thursday, February 9, 2017 10:31 AM

Def. Let $\mathbb{G} \subseteq GL_n(\mathbb{C})$ be a Zariski-closed subgroup defined over $\mathbb{R}$. Suppose $G := \mathbb{G}(\mathbb{R}) \subseteq GL_n(\mathbb{R})$ is Zariski-dense in $\mathbb{G}$.

We say G is semisimple if it does NOT have an infinite abelian normal subgroup.

. Ex. $SL_n(\mathbb{R})$ or $\prod_{i=1}^{k} SL_{n_i}(\mathbb{R})$ are semisimple Lie groups.

Fact. Any semisimple group G is an almost product of almost simple Lie groups, i.e. $G = G_1 \cdot G_2 \cdot \cdots \cdot G_k$ where

$|Z(G_i)| < \infty$ and $G_i / Z(G_i)$ has no proper normal subgroup;

$G_i \cap \prod_{j \neq i} G_j \subseteq Z(G_i)$ and so finite.

Fact G° is of finite-index in G; and so $G^\circ = [G^\circ, G^\circ]$.

So $G^\circ \hookrightarrow SL_n(\mathbb{R})$.

Fact (Mostow) there is an embedding $G^\circ \hookrightarrow SL_n(\mathbb{R})$ s.t.

$\forall g \in G^\circ$, $g^t \in G^\circ$.

From this point we will assume:

$G \subseteq SL_n(\mathbb{R})$, connected, semisimple without compact factor and $g \in G \Rightarrow g^t \in G$.

Lecture 11: Two models of X and the action of G

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Let $K := G \cap O(n)$ where $O(n) := \{ g \in GL_n(\mathbb{R}) \mid gg^t = I \}$;

Let $X := G \cap P(n)$ where $P(n) := \{ g \in GL_n(\mathbb{R}) \mid g \text{ is positive-definite} \}$

- $g \in GL_n(\mathbb{R}) \Rightarrow (gg^t)^{1/2} \in P(n)$ and $(gg^t)^{-1/2} g \in O(n)$.
- $a \in G \cap P(n) \overset{?}{\Rightarrow}$ the 1-parameter group $s \mapsto a^s$ is in G.

<u>Lemma</u> $G = X \cdot K$ and $X \cap K = \{I\}$.

<u>Pf.</u>

$$\left.\begin{array}{l} g = (gg^t)^{1/2} \cdot \left((gg^t)^{-1} g \right) \\ gg^t \in G \cap P(n) \Rightarrow (gg^t)^{1/2} \in G \end{array}\right\} \Rightarrow G = X \cdot K.$$

$$\left.\begin{array}{l} g \in X \cap K \Rightarrow I = g \cdot g^t = g^2 \\ g \text{ is positive-definite} \end{array}\right\} \Rightarrow g = I. \qquad \square$$

<u>Cor.</u> $G/K \to X$, $gK \mapsto gg^t$ is a homeomorphism.

<u>Pf.</u> <u>Well-defined</u>. $g_1 K = g_2 K \Rightarrow \exists k \in K,\ g_1 = g_2 k$

$$\Rightarrow g_1 g_1^t = g_2 k k^t g_2^t = g_2 g_2^t.$$

<u>1-1</u>. $g_1 g_1^t = g_2 g_2^t \Rightarrow g_1^{-1} g_2 g_2^t (g_1^t)^{-1} = I$

$$\Rightarrow g_1^{-1} g_2 \in K \Rightarrow g_1 K = g_2 K.$$

<u>onto</u>. $g \in X \Rightarrow g^{1/2} \in G$ and $g^{1/2} K \mapsto g^{1/2} g^{1/2} = g$.

It is clearly continuous; The inverse map is $x \mapsto x^{1/2} K$ which is continuous. $\blacksquare$

Lecture 11: Two models of and a metric on X

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- $G \curvearrowright G/K$ by left multiplication; we define $G \curvearrowright X$ in a way that makes $G/K \longrightarrow X$, $gK \longmapsto gg^t$, G-equivariant.

$$g \cdot x := g x g^t \quad \Big(\text{Notice } g(g'K) = (gg')K \longmapsto (gg')(gg')^t = g(g'g'^t)g^t. \Big)$$

- $\mathcal{P}(n) \underset{\exp}{\overset{\log}{\rightleftarrows}} S(n) := \{ x \in M_n(\mathbb{R}) \mid x = x^t \}$ are analytic homeom.

So $\log : X \longrightarrow \mathfrak{p} \subseteq \mathrm{Lie}(G)$ is a bianalytic homeomorphism.

<u>Def.</u> On $\mathcal{P}(n)$ we define the following Riemannian metric:

$$\left(\frac{ds}{dt}\right)^2 = \mathrm{Tr}\left((p^{-1} \dot{p})^2 \right),$$

where $p(t)$ is a differentiable path in $\mathcal{P}(n)$.

Equivalently we can identify the tangent space $T_p X$ with the symmetric spaces and define the Riemannian dot product as $\langle x_1, x_2 \rangle_p := \mathrm{tr}(p^{-1} x_1 p^{-1} x_2)$. Notice that $\langle x_1, x_2 \rangle_p$ is clearly bilinear, and $\langle x, x \rangle_p = \mathrm{tr}(p^{-1} x p^{-1} x)$

$$= \mathrm{tr}(p^{-1/2} x p^{-1/2} \cdot p^{-1/2} x p^{-1/2})$$

and $(p^{-1/2} x p^{-1/2})^t = p^{-1/2} x p^{-1/2}$. So $\langle x_1, x_2 \rangle_p$ is positive definite.

<u>Lemma</u>. $G \curvearrowright X$ preserves this Riemannian structure.

Lecture 11: Why is X symmetric?

Tuesday, February 14, 2017 8:49 AM

Pf. $\langle g\cdot x_1, g\cdot x_2\rangle_{g\cdot p} = tr((gpg^t)^{-1}(gx_1g^t)(gpg^t)^{-1}(gx_2g^t))$

$$= tr((g^t)^{-1} p^{-1} \cancel{g^{-1}} \cancel{g} x_1 \cancel{g^t} \cancel{(g^t)^{-1}} p^{-1} \cancel{g^{-1}} \cancel{g} x_2 g^t)$$

$$= tr((g^t)^{-1} p^{-1} x_1 p^{-1} x_2 (g^t))$$

$$= tr(p^{-1} x_1 p^{-1} x_2) = \langle x_1, x_2\rangle_p . \qquad \blacksquare$$

Lemma (Symmetric) $\forall\, p \in X$, $\exists\, \sigma_p \in \mathrm{Isom}(X)$, $\sigma_p(p) = p$ and $(d\sigma_p)_p(x) = -x$.

Pf. Since $G \curvearrowright X$ transitively and isometrically, it is enough to prove this for $I \in X$.

For G/K model, let $\sigma_I(gK) = (g^t)^{-1}K$ (why is it well-defined?)

For $\mathcal{P}(n)$ model, let $\sigma_I(p) = p^{-1}$. $\square$

So X is a symmetric Riemannian space.

The following lemma helps us to get a better understanding of geodesics in X.

Theorem. Along any differentiable path $p(t)$ in $\mathcal{P}(n)$,

$$\mathrm{Tr}\left(\left(\frac{d}{dt}\log p(t)\right)^2\right) \le \mathrm{Tr}\left((p^{-1}\dot{p})^2\right),$$

with equality if and only if p and $\dot{p}$ commute.