Lecture 17, 18: Γ- equivariant QI between symmetric spaces

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Pf. Since Γ is torsion-free, it acts freely on X_i's. So

][]y

$M_i := \Gamma \backslash X_i$ are compact manifolds. So they can be triang.

and are finite simplicial complexes $\Rightarrow$

$$\exists\ \phi : X_1 \to X_2 \quad \text{which is } \Gamma\text{-equivariant (continuous)}$$

$$\overline{\Phi} : M_1 \to M_2 \quad \text{is simplicial.}$$

Since any simplicial map satisfies the Lipschitz condition, we get

$$\exists \lambda > 1, \quad d(\overline{\Phi}(\bar{x}), \overline{\Phi}(\bar{y})) \le \lambda\, d(\bar{x}, \bar{y}).$$

So for any curve $c : [0,1] \to M_1$ with finite length we have

$$\ell(\overline{\Phi}(c)) = \sup_{0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1} \sum d(\overline{\Phi}(c(t_i)), \overline{\Phi}(c(t_{i+1})))$$

$$\le \lambda \sup_{t_i} \sum d(c(t_i), c(t_{i+1}))$$

$$= \lambda\, \ell(c).$$

Since the metric on X_i is the covering metric $X_i \xrightarrow{\pi_i} M_i$,

we have: $\forall x, y \in X_i$, $d(x,y) = \inf\limits_{\substack{c : [0,1] \to X_i \\ c(0) = x,\ c(1) = y}} \ell(\pi_i(c))$.

$$\Rightarrow d(\phi(x), \phi(y)) = \inf_{\substack{c : [0,1] \to X_2 \\ c(0) = \phi(x),\ c(1) = \phi(y)}} \ell(\pi_2(c)) \le \inf_{\substack{c : [0,1] \to X_1 \\ c(0) = x,\ c(1) = y}} \ell(\pi_2(\phi(c)))$$

Lecture 18: Γ-equivariant QI

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$$= \inf_{\substack{c:[0,1]\to X_1 \\ c(0)=x,\, c(1)=y}} \ell(\overline{\Phi}(\pi_1(c)))$$

$$\leq \lambda \inf_{\substack{c:[0,1]\to X_1 \\ c(0)=x,\, c(1)=y}} \ell(\pi_1(c)) = \lambda\, d(x,y).$$

- $d(x,y) \leq d(y, \gamma_0 y) + d(\gamma_0 y, x)$ where $d(\gamma_0 y, x) \leq \operatorname{diam} M_1$

$$d(\Phi(x), \Phi(\gamma_0 y)) \leq \lambda\, d(x, \gamma_0 y) \leq \lambda \operatorname{diam} M_1.$$

- $d(\Phi(x), \Phi(y)) \geq d(\Phi(y), \gamma_0 \Phi(y)) - \lambda \operatorname{diam} M_1$

Švarc–Milnor' lemma

$$\geq \lambda'^{-1}\, d_\Omega(e, \gamma_0) - C'$$

$$\geq \lambda''^{-1}\, d(y, \gamma_0 y) - C'' \geq \lambda''^{-1} d(x,y) - C'''.$$

- We get the quasi-surjectivity again Švarc–Milnor's lemma and a single orbit. ∎

Lecture 18: Uniform injectivity radius

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- A closed curve in M up to homotopy determines a conjugacy class $C(\gamma_0)$ of Γ. And the infimum of the length of these curves is $d_{\gamma_0} := \inf_{x \in X} d(x, \gamma_0 x)$, the displacement of Γ.

Since Γ is a cocompact lattice, all of its elements are semisimple. So $d_{\gamma_0} > 0$, as Γ has no torsion-element.

- Let $S := \{ c \subseteq M_1 \mid$ ① c is a closed curve, ② $l(c) \leq l_0$, ③ c is homotopically non-trivial $\}$.

Since M_1 is compact, S is compact with respect to the Chabauty topology on closed subsets of M_1: (Hausdorff distance topology). So there is a smallest length curve among elements of S. Hence $\inf_{\gamma \in \Gamma} d_\gamma = d_{\gamma_0} = r_0 > 0$ for some $\gamma_0 \in \Gamma$.

- So any point of M_i has an injectivity radius of $\geq r_0/2$, ie $\forall x \in X_i$, $\pi_i|_{B(x, r_0/2)}$ is injective.

Lecture 18: Space of flats

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Let $\mathcal{F}_i$ be the space of maximal flats of X_i; with the topology of uniform convergence on compact sets. Our next goal is to show that ϕ induces a homeomorphism $\Phi: \mathcal{F}_1 \to \mathcal{F}_2$.

To define the Φ we will prove:

<u>Proposition</u> ① $\forall\ F_1 \in \mathcal{F}_1$, $\exists!\ F_2 \in \mathcal{F}_2$ s.t.

$$hd(\phi(F_1), F_2) \ll_{\phi} 1.$$

② Define $\Phi(F_1) \in \mathcal{F}_2$ s.t.

$$hd(\phi(F_1), \Phi(F_1)) <_{\phi} 1.$$

Then Φ is a homeomorphism and it is clearly Γ-equivariant.

. We start by a subset of $\mathcal{F}_1$:

$$\mathcal{F}_{1,\Gamma} := \{ F \in \mathcal{F}_1 \mid F \text{ is } \Gamma\text{-compact} \text{ i.e. } {}_{\Gamma}\backslash \Gamma F \text{ is compact} \}.$$

<u>Lemma</u>. $\mathcal{F}_{\Gamma}$ is dense in $\mathcal{F}$.

<u>Pf</u>. $F \in \mathcal{F} \Rightarrow F = A x_0$ where A is a maximal polar subgp. Suppose $a \in A_t$, i.e. $\forall \alpha \in \Delta$, $\alpha(a) > t$ where Δ is a set of simple roots of A.

Lecture 18: Γ-compact flats are dense in the space of flats

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We proved the following Proposition for $SL_n(\mathbb{R})$. The same argument works in general.

Proposition. $\forall$ nbhd $\mathcal{O}_G$ of I in G,
$\forall$ nbhd $\mathcal{O}_A$ of I in A,
$t>1$, $\exists$ $U_{G'}$ a nbhd of I in G, s.t.

$$\forall a\in A_t,\ UaU \subseteq \{g\, a a' m\, g^{-1} \mid a'\in \mathcal{O}_A,\ g\in\mathcal{O}_G,\ m\in M\}$$

where M is the maximum compact subgroup of $C_G(A)$.

Choose $\mathcal{O}_A$ small enough, e.g. $\{a'\in A \mid \alpha(a')\le \sqrt{t}$ for any $\alpha\in\Delta\}$, s.t. $A_t\mathcal{O}_A \subseteq A_{t'}$ for some $t'>1$, e.g. for the above choice we get $A_t\mathcal{O}_A\subseteq A_{\sqrt{t}}$.

Let U_G be a nbhd of I which is given by the above proposition for an arbitrarily small nbhd $\mathcal{O}_G$ and $\mathcal{O}_A$ as above.

By Selberg's argument, $\exists\, 0<n$ s.t $U_G a^n U_G \cap \Gamma \neq \emptyset$.

Let $\gamma\in\Gamma\cap U_G a^n U_G$. Hence $\exists\, g\in\mathcal{O}_G$, $a'\in A_{\sqrt{t}}$, $m\in M\subseteq C_G(A)$

$$\gamma = g\, a' m\, g^{-1}.$$

Lecture 18: Γ- compact flats are dense in the space of flats

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So $\mathrm{pol}(\gamma) = g\, a\, g^{-1}$ is regular, and

$$C_G(\mathrm{pol}(\gamma)) = C_G(g\, a\, g^{-1}) = g\, C_G(a)\, g^{-1}$$
$$= g\, C_G(A)\, g^{-1} \supseteq g\, A\, g^{-1}.$$

- By another Selberg's lemma (that we proved earlier) we have that $C_\Gamma(\gamma)$ is a cocompact lattice in $C_G(\gamma)$.
- On the other hand, $C_G(A) = M \cdot A$ (see below), and so

$$C_G(\gamma) = g\,(C_M(m)\, A)\, g^{-1}.$$

- Therefore $\Gamma \cap M' A^g \backslash M' A^g$ is compact where $M' := C_M(m)$ and $A^g := g A g^{-1}$.

Claim If A is a maximal polar subgroup, then

① $C_G(A) = M A$ where M is the maximal compact subgroup

② If $F = A \cdot x$ is a flat, then $M \subseteq \mathrm{Stab}(x)$. So $F = C_G(A) \cdot x$.

Pf of claim ① After conjugating A, we can assume that A is a maximal abelian subgroup of $\mathcal{P} := \mathcal{P}(n) \cap G$. So $C_G(A)$ is self-adjoint and $C_G(A) = C_K(A) \cdot (C_G(A) \cap \mathcal{P}(n)) = C_K(A) \cdot A$.

Lecture 18: Γ- compact flats are dense in the space of flats

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Since $A \cap K = 1$, we get $C_G(A) \simeq C_K(A) \times A$. And so $M := C_K(A)$ is the maximal compact subgroup of $C_G(A)$.

② After moving x and conjugating A, we can assume $x = x_K$. So $F = A \cdot x_K$ is a flat which passes through x_K. Hence $A \subseteq P := G \cap P(n)$. So, by part 1, $M \leq K$, which implies $M \cdot x_K = x_K$ and $C_G(A) \cdot x_K = x_K$.

Since $F = A \cdot x_0$ is a maximal flat, $F = C_G(A) \cdot x_0$ and $M \cdot x_0 = x_0$.

Hence $F_\gamma := C_G(\gamma)\, g x_0 = (g\, C_M(m)\, A\, g^{-1}) g x_0$

$= g\, A\, C_M(m)\, x_0 = gF$ is a flag; and

Since $C_\Gamma(\gamma) \backslash C_G(\gamma)$ is compact, $\Gamma \backslash \Gamma F_\gamma$ is compact.

So $F_\gamma = gF \in \mathcal{F}_\Gamma$ for some $g \in O_G$. Since O_G is an arbitrary nbhd of I, we get the desired result. ■