Lecture 19: Equality of $\mathbb{R}$-ranks

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<u>Proposition</u>. In the setting of Mostow's Strong Rigidity Theorem,

① $\mathbb{R}$-rank of G_1 = $\mathbb{R}$-rank of G_2.

② For $F_1 \in \mathcal{F}_{1,\Gamma}$, let $\Delta_{F_1} := \Gamma \cap G_{F_1}$, where $G_{F_1} := \{g \in G \mid g F_1 = F_1\}$. Then $\exists!\, F_2 \in \mathcal{F}_{2,\Gamma'}$ s.t. $\theta(\Delta_{F_1}) \subseteq G_{F_2}$ and $\theta(\Delta_F)$ is a cocompact lattice in G_{F_2}.

To prove the above result, we start with understanding $\mathbb{R}$-rank of G in terms of group theoretic properties of Γ.

<u>Lemma</u>. $\mathbb{R}$-rank of $G = \max\{\operatorname{rank}(\Delta) \mid \Delta \subseteq \Gamma \text{ is a free abelian group}\}$.

<u>Pf.</u> $(\geq)$ Let $\Delta \subseteq \Gamma$ be a free abelian group.

Since Γ is cocompact, it consists of semisimple elements. Since Δ is abelian, Δ is diagonalizable over $\mathbb{C}$. Hence the Zariski-closure $\mathbb{T}$ of Δ is diagonalizable over $\mathbb{C}$, and $\Delta \subseteq T := \mathbb{T}(\mathbb{R})$. Hence T can be written as a product of a polar subgroup A and a compact abelian group C.

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Hence Δ is a discrete subgp of $A \times C$. Since C is compact, the projection of Δ to the A-part is still discrete. Moreover, since Δ is discrete and torsion-free, $\Delta \cap C = 1$. So $pr_A: A \times C \to A$ induces an isomorphism $\Delta \to pr_A(\Delta)$. Since $A \simeq \mathbb{R}^{r_0}$ and $r_0 \leq \mathbb{R}$-rank of G, we get that

$$\text{rank}(\Delta) = \text{rank}(pr_A(\Delta)) \leq r_0 \leq \mathbb{R}\text{-rank of } G.$$

$(\leq)$ In the previous lecture, we proved $\mathcal{F}_\Gamma$ is dense in $\mathcal{F}$. Let $F \in \mathcal{F}_\Gamma$. So $\Gamma \cap G_F$ is a cocompact lattice in G_F. And $G_F = MA$ where M is the maximal compact subgp of $C_G(A)$, $F = Ax_0$, A is a maximal polar subgp, and $Mx_0 = x_0$. Since $A \cap M = 1$ and $M \subseteq C_G(A)$, we get $G_F \simeq A \times M$. So $\Delta := \Gamma \cap G_F \hookrightarrow A \times M$ as a cocompact lattice. Since M is compact, $pr_A(\Delta)$ is a lattice in A; Since Γ is discrete and torsion-free, $M \cap \Gamma = 1$. So $\Delta \simeq pr_A(\Delta) \subseteq A$ a lattice in A. Hence Δ is a free abelian gp and $\text{rank}(\Delta) = \mathbb{R}$-rank of G. ∎

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Lemma implies the equality of $\mathbb{R}$-ranks which is part 1 of Proposi.

To get the second part, it is enough to show the following:

Lemma. Suppose $\Delta \subseteq \Gamma$ is a free abelian group whose rank is the $\mathbb{R}$-rank of G. Then there is a unique flat F s.t. $\Delta \subseteq G_F$. Moreover Δ is a cocompact lattice in G_F.

Pf. As in the proof of previous lemma, $\Delta \subseteq T = A \cdot C$ where A is a polar subgroup and C is a compact abelian group. Since Δ is discrete and torsion-free and C is compact, $\Delta \simeq pr_A(\Delta) \hookrightarrow A$ is a discrete subgp. So $\mathrm{rank}(\Delta) \leq \dim A$. Since $\dim(A) \leq \mathbb{R}$-rank of $G = \mathrm{ran}(\Delta)$, we get that A is a max. polar subgp. And $pr_A(\Delta)$ is a cocompact lattice in A. Therefore Δ is a cocompact lattice in $A \cdot C$.

$\exists g \in G$ s.t. $gAg^{-1} \subseteq P := P(n) \cap G$ and $gCg^{-1} \subseteq K := O(n) \cap G$.

$F := AC\, g^{-1} x_K = g^{-1} \cdot gAg^{-1} \cdot gCg^{-1} \cdot x_K = g^{-1}(gAg^{-1})\, x_K$ is a flat.

And $\Delta \subseteq AC \subseteq G_F$. Moreover G_F/AC is compact $\Rightarrow \Delta$ is cocomp. in G_F. Exercise show uniqueness. ∎

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Pf of proposition. ① $\mathbb{R}$-rank of $G_1 = \max\{\operatorname{rank}(\Delta) \mid \Delta \subseteq \Gamma \text{ free abelian}\}$

$= \mathbb{R}$-rank of G_2.

② $F_1 \in \mathcal{F}_{1,\Gamma} \Rightarrow \Delta_1 := \Gamma \cap G_{F_1}$ is a lattice in G_{F_1}

and $\operatorname{rank}(\Delta_1) = \mathbb{R}$-rank of G_1

$\Rightarrow \operatorname{rank}(\theta(\Delta_1)) = \mathbb{R}$-rank of $G_1 = \mathbb{R}$-rank of G_2

(by the 2nd lemma) $\Rightarrow \exists!\ F_2 \in \mathcal{F}_{2,\Gamma}$, $\theta(\Delta_1) \subseteq G_{F_2}$. ■

Lemma. $\forall F_1 \in \mathcal{F}_{1,\Gamma}$, let $F_2 \in \mathcal{F}_{2,\Gamma}$ be given by Proposition.

Then $hd(\Phi(F_1), F_2) \ll_\Phi 1$.

Pf. Step 1. Let $pr_i : X_i \longrightarrow F_i$ be the orthogonal projection.

Then $pr_2(\Phi(F_1)) = F_2$.

Step 2. $pr_1(\Phi^{-1}(N_{c'}(F_2))) = F_1$ where c' depends on the QI parameters of Φ.

Step 3. Finishing the proof.

$\boxed{\text{Step 1}}$ Let $\Delta = \Gamma \cap G_{F_1}$. Then, as we showed earlier, Δ is a free abelian group, and $\Delta \backslash F_1$ is compact. And the way F_2

F_2 is defined, we have $\theta(\Delta) \subseteq G_{F_2}$ and $\theta(\Delta)\backslash F_2$ is compact.

Let $\psi : F_1 \longrightarrow F_2$ be $\psi(x) = pr_2(\phi(x))$. Then,

for any $\gamma \in \Delta$, $\psi(\gamma x) = pr_2(\phi(\gamma x))$

$$= pr_2(\gamma \phi(x))$$

$$\overset{\boxed{\theta(\gamma) \in G_{F_2}}}{=} \theta(\gamma) \cdot pr_2(\phi(x))$$

. Since F_i's are contractible, we get that

$$\overline{\psi} : \Delta\backslash F_1 \longrightarrow \theta(\Delta)\backslash F_2 \text{ is a homotopy equivalence.}$$

So $\overline{\psi}$ induces isomorphism of the homotopy groups $H_r(\Delta\backslash F_1)$ and $H_r(\theta(\Delta)\backslash F_2)$. Since these are r_0-dim. tori ($r_0 = \mathbb{R}$-rank of G_i), the top dimensional cycle should be mapped to the top dimensional cycle $\Rightarrow \overline{\psi}$ is onto $\Rightarrow \psi$ is onto.

$\boxed{\text{Step 2}}$ $\phi : X_1 \longrightarrow X_2$ is (λ, C)-QI, and λ-Lipschitz. So

$$\lambda\, d(x,y) \geq d(\phi(x), \phi(y)) \geq \lambda^{-1} d(x,y) - C.$$ Hence, if

$$d(x,y) \geq \underbrace{2\lambda C}_{b},$$ then

$$d(\phi(x), \phi(y)) \geq \lambda^{-1} d(x,y) - C \geq \lambda^{-1} d(x,y) - \frac{\lambda^{-1}}{2} d(x,y)$$

$$\geq \underbrace{(2\lambda)^{-1}}_{k} d(x,y).$$

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<u>Claim</u>. $pr_1(\Phi^{-1}(N_b(F_2))) = F_1$.

<u>Pf</u>. Let $\Delta \subseteq G_{F_1}$ be a free abelian group of rank r_0. (So $\Delta\backslash F_1$ and $\theta(\Delta)\backslash F_2$ are tori.)

The idea is to construct $\xi : \theta(\Delta)\backslash F_2 \to \Delta\backslash X_1$ in a way that it induces an isomorphism between $H_{r_0}(\theta(\Delta)\backslash F_2)$ and $H_{r_0}(\Delta\backslash X_1)$.

On the other hand, the orthogonal projection pr_1 induces an isomorphism between $H_{r_0}(\Delta\backslash X_1)$ and $H_{r_0}(\Delta\backslash F_1)$ (X_1 and F_1 are contractible and pr_1 is Δ-equivariant. Hence $pr_1 \circ \xi$ induces a bijection between $H_{r_0}(\theta(\Delta)\backslash F_2)$ and $H_{r_0}(\Delta\backslash F_1)$. Therefore

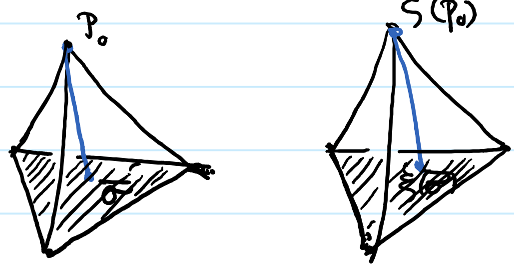
$$pr_1 \circ \xi(\theta(\Delta)\backslash F_2) = \Delta\backslash F_1 .$$

Now, it would be enough to make sure $\xi(F_2)$ is in $N_b(\Phi^{-1}(F_2))$.

. To construct ξ, we start with a triangularization Σ of $\theta(\Delta)\backslash F_2$, s.t. $\forall$ simplicial $\sigma \in \Sigma$, $\operatorname{diam}(\sigma) \le b/k$. We will define ξ on the vertices Σ_0 of Σ and then we extend ξ on larger simplicials as follows: $\forall \sigma \in \Sigma_d$, let P_0 be a vertex of σ and $\bar{\sigma}$ be the face of

σ which is in front of p_0. Then, for any $p \in \overline{\sigma}$, we send

$[p_0, p]$ to $[\xi(p_0), \xi(p)]$.



For any $p \in \Sigma_{i_0}$, let $\xi(p) \in {}_\Delta X_1$ be st. $\Phi_\Delta(\xi(p)) = p$.

(why is there such $\xi(p)$? This is non-trivial.)

Claim 1 $\forall p_1, p_2 \in \sigma \cap \Sigma_{i_0}$, $d(\xi(p_1), \xi(p_2)) \le b$.

Pf. If not, $d(\Phi_\Delta(\xi(p_1)), \Phi_\Delta(\xi(p_2))) > \frac{b}{k}$.

$$\underbrace{d(\Phi_\Delta(\xi(p_1)), \Phi_\Delta(\xi(p_2)))}_{= d(p_1, p_2)}$$

Claim 2 $\forall \sigma \in \Sigma$, $d(\Phi_\Delta(\xi(\sigma)), \sigma) \le kb$.

Pf. $\forall p \in \sigma$, $\xi(p)$ is in the simplicial with vertices $\{\xi(p_i)\}$

where p_i's are vertices of σ. So $d(\xi(p_i), \xi(p)) \le b$. Hence

$d(\Phi_\Delta(\xi(p_i)), \Phi_\Delta(\xi(p))) \le bk$, which implies

$d(p_i, \Phi_\Delta(\xi(p))) \le bk \Rightarrow d(\sigma, \Phi_\Delta(\xi(\sigma))) \le bk$.

Claim 3 $\forall p \in {}_{\theta(\Delta)}\backslash F_2$, $d(\Phi_\Delta(\xi(p)), p) \le 2kb$.

Pf. $p \in \sigma$ for some $\sigma \in \Sigma \Rightarrow \exists q \in \sigma$ st. $d(\Phi_\Delta(\xi(p)), q) \le kb$

And $d(p, q) \le b/k$. So $d(\Phi_\Delta(\xi(p)), p) \le 2kb$.

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Going to a subgroup of finite-index of Δ we can make sure that $\forall p' \in \Delta\backslash X_1$ has injectivity radius $\geq 4kb$.

So $\Phi_\Delta \circ \xi$ can be deformed to the identity map along the shortest pass connecting $\Phi_\Delta \circ \xi(p)$ to p. So

$$\begin{tikzcd}
\Delta\backslash F_2 \arrow[r, "\xi"] \arrow[d, equal] & \Delta\backslash X_1 \arrow[r, "\Phi_\Delta"] & \Delta\backslash X_2 \\
\Delta\backslash F_2 \arrow[rr] & & \Delta\backslash F_2 \arrow[u, phantom, "\cup" description]
\end{tikzcd}$$

induces homomorphisms

$$\begin{tikzcd}
H_{r_0}(\Delta\backslash F_2) \arrow[r] & H_{r_0}(\Delta\backslash X_1) \arrow[r] & H_{r_0}(\Delta\backslash X_2) \\
& \arrow[r, phantom, "\circlearrowleft"] & H_{r_0}(\Delta\backslash F_2) \arrow[u, "S"'] \arrow[llu, equal]
\end{tikzcd}$$

$\Rightarrow H_{r_0}(\Delta\backslash F_2) \xrightarrow{\xi} H_{r_0}(\Delta\backslash X_1)$ is an isomorphism.

On the other hand the orthogonal projection Pr_{F_2} is Δ-equivari.

And so it induces an isomorphism $H_{r_0}(\Delta\backslash X_1) \xrightarrow{\sim} H_{r_0}(\Delta\backslash F_1)$.

Therefore $H_{r_0}(\Delta\backslash F_2) \xrightarrow{Pr_{F_1}\circ\xi} H_{r_0}(\Delta\backslash F_1)$ is an isomorphism.

Thus $Pr_{F_1}(\xi(\Delta\backslash F_2)) = \Delta\backslash F_1 \Rightarrow Pr_{F_1}(N_b(\Phi^{-1}(F_2))) = F_1$.