Lecture 21: Splices and their description

Tuesday, March 14, 2017 10:58 AM

The Key Geometric Ingredients:

For two maximal flats F_0 and F and $x \in F$, let

$$F \underset{x}{\rightsquigarrow} F_0 := \bigcup \left\{ r \;\middle|\; \begin{array}{l} r \text{ is ray in } F \\ r\text{'s orgin is } x \\ r \subseteq N_{O(1)}(F_0) \end{array} \right\}.$$

If $F \underset{x}{\rightsquigarrow} F_0 \neq \emptyset$, then for $s > d(x, F_0)$

Theorem ① $F \underset{x}{\rightsquigarrow} F_0 \subseteq N_s(F_0) \cap F \subseteq N_{O(s)}(F \underset{x}{\rightsquigarrow} F_0)$.

② $F \underset{x}{\rightsquigarrow} F_0$ is a convex hull of a finite union of chambers and chamber walls in F of origin x.

Def. • $F \underset{x}{\rightsquigarrow} F_0$ is called a splice.

• A splice C is called irreducible if

$$hd(C, \bigcup_{i=1}^{m} C_i) < \infty \Rightarrow hd(C, C_{i_0}) < \infty \text{ for some } i_0,$$

where C_i's are splices.

Lemma. A splice is irreducible $\Leftrightarrow$ it is a closed face of a Weyl chamber. (it can be the entire Weyl chamber).

Definition. For two subsets Y_1 and Y_2 of a metric space, we say $Y_1 < Y_2$ if $Y_1 \subseteq N_{O(1)}(Y_2)$, and $Y_1 \sim Y_2$ if $hd(Y_1, Y_2) \ll 1$.

Lecture 21: Intersection of nbhds of maximal flats

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It is clear that $\sim$ defines an equivalence relation. The mentioned geometric ingredient implies that

$$F \underset{x}{\rightsquigarrow} F_0 \sim N_{d(x,F_0)^+}(F_0) \cap F$$

if non-empty.

<u>Corollary</u>. For any r_1, r_2, s_1, s_2 and two flats F_1 and F_2,

$$N_{r_1}(F_1) \cap N_{r_2}(F_2) \sim N_{s_1}(F_1) \cap N_{s_2}(F_2)$$

if non-empty.

<u>Pf</u>. <u>Step 1</u>. $N_{r_1}(F_1) \cap N_{r_2}(F_2) \subseteq N_{r_2}(N_{r_1+r_2}(F_1) \cap F_2)$.

<u>Pf of step 1</u>. $d(x, F_i) < r_i \Rightarrow d(x, pr_{F_i}(x)) < r_i \Rightarrow d(pr_{F_1}(x_1), pr_{F_2}(x_2))$ is less than r_1+r_2. So $pr_{F_2}(x_2) \in N_{r_1+r_2}(F_1) \cap F_2$. Therefore

$$x \in N_{r_2}(N_{r_1+r_2}(F_1) \cap F_2).$$

<u>Step 2</u>. $N_{r_1+r_2}(F_1) \cap F_2 \subseteq N_{r_2}(N_{r_1}(F_1) \cap N_{r_2}(F_2))$.

<u>Pf</u>. $x_2 \in N_{r_1+r_2}(F_1) \cap F_2 \Rightarrow d(x_2, pr_{F_1} x_2) < r_1 + r_2$. Let $y \in [x_2, pr_{F_1} x_2]$ s.t. $d(y, x_2)/d(x_2, pr_{F_1} x_2) < r_2/r_1+r_2$ (if $x_2 \in F_1$, we are done).

$$\Rightarrow d(y, x_i) < r_i \Rightarrow \left.\begin{array}{l} y \in N_{r_1}(F_1) \cap N_{r_2}(F_2) \\ d(y, x_2) < r_2 \end{array}\right\} \Rightarrow x_2 \in N_{r_2}(N_{r_1}(F_1) \cap N_{r_2}(F_2)).$$

Hence $N_{r_1}(F_1) \cap N_{r_2}(F_2) \sim N_{r_1+r_2}(F_1) \cap F_2$ and if one of them is non-empty so is the other one.

Lecture 21: Sets of finite Hausdorff distance under a QI

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So it is enough to show

Step 3. $N_r(F_1)\cap F_2 \sim N_s(F_1)\cap F_2$ if non-empty.

Pf of step 3. Suppose $x \in N_s(F_1)\cap F_2 \cap N_r(F_1)$. Then by the key geometric ingredient we have:

$$N_s(F_1)\cap F_2 \sim F_2 \underset{x}{\rightsquigarrow} F_1 \sim N_r(F_1)\cap F_2$$

if $F_2 \underset{x}{\rightsquigarrow} F_1 \neq \emptyset$.

If $F_2 \underset{x}{\rightsquigarrow} F_1 = \emptyset$, by convexity of $N_s(F_1)\cap F_2$ and compactness of space of directions we get that

$N_s(F_1)\cap F_2$ and $N_r(F_1)\cap F_2$ are bounded. ■

$\boxed{\text{Observation}}$ Φ is a QI $\Rightarrow$ • $Y_1 \sim Y_1'$ if and only if $\Phi(Y_1)\sim\Phi(Y_1')$.

$\Phi: X_1 \to X_2$

• $\Phi^{-1}(\Phi(Y))\sim Y$

and the implied constants depend only on the QI parameters.

Lecture 21: Map on the set of the classes of splices

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<u>Proposition</u> Suppose $S := F_1 \overset{}{\underset{x_1}{\rightsquigarrow}} F_1'$ is a splice in X_1. Then

$$\phi(F_1 \underset{x_1}{\rightsquigarrow} F_1') \sim \overline{\Phi}(F_1) \underset{x_2}{\rightsquigarrow} \overline{\Phi}(F_2)$$

for any $x_2 \in \overline{\Phi}(F_1)$.

<u>Pf.</u> $\phi(F_1 \underset{x_1}{\rightsquigarrow} F_1') \sim \phi(N_{r_1}(F_1) \cap F_2)$ if $r_1 > d(F_1', x_1)$

$$\sim \phi(N_{r_1+O_{\lambda C}(1)}(F_1) \cap N_{r_2+O_{\lambda C}(1)}(F_2))$$

$$\sim \phi(\phi^{-1}(\phi(N_{r_1}(F_1)) \cap \phi^{-1}(\phi(N_{r_2}(F_2))))$$

$$= \phi(N_{r_1}(F_1)) \cap \phi(N_{r_2}(F_2))$$

Since ϕ is surjective

$$N_{\lambda^{-1}r-C}(\phi(Y)) \subseteq \phi(N_r(Y)) \subseteq N_{\lambda r+C}(\phi(Y)).$$

$$N_{\lambda^{-1}r-C-O_{\lambda C}(1)}(\overline{\Phi}(F)) \subseteq \phi(N_r(F)) \subseteq N_{\lambda r+C+O_{\lambda, C}(1)}(\overline{\Phi}(F))$$

for any flat F.

by and the previous corollary

$$\sim N_{O(r_1)}(\overline{\Phi}(F_1)) \cap N_{O(r_2)}(\overline{\Phi}(F_2))$$

for $r_1, r_2 \gg 1$.

$$\sim \overline{\Phi}(F_1) \underset{x_2}{\rightsquigarrow} \overline{\Phi}(F_2)$$

for any $x_2 \in \overline{\Phi}(F_1)$. ∎

Lecture 21: Map on the set of the classes of splices

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Let $\mathcal{S}(X) := \{ [F \underset{x}{\rightsquigarrow} F'] \mid F, F' \in \mathcal{F}_X \}$.

<u>Corollary</u>. $\exists$ a Γ-equivariant bijection $\phi^* : \mathcal{S}(X_1) \to \mathcal{S}(X_2)$.

s.t. $\phi^*(S) = [\phi(S)]$.

<u>Pf</u>. Let $\phi^*([F_1 \underset{x}{\rightsquigarrow} F_1']) := [\Phi(F_1) \xrightarrow[\mathrm{pr}_{\Phi(F_1)}(\phi(x))]{} \Phi(F_1')]$.

It is easy to see that ϕ^* is Γ-equivariant.

By the previous proposition we have $[\phi(S)] = \phi^*([S])$

for any splice S.

Since Φ is a bijection and $[F_1 \underset{x}{\rightsquigarrow} F_1'] = [F_1 \underset{x'}{\rightsquigarrow} F_1']$

for any $x, x' \in F_1$, one can see that ϕ^* is a

bijection. ∎

<u>Lemma</u>. Suppose S and S' are two splices, and $S \sim S'$.

Then S irreducible $\Rightarrow$ S' irreducible.

Lecture 21: Map on the maximal boundary

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dim.

<u>Claim</u> $\Phi^*(\xi)$ is an irreducible splice class.

<u>Pf of claim.</u> Let S' be a splice in $\Phi^*(\xi)$. If it is not irreducible, $S' \sim S_1' \cup \dots \cup S_n'$ and $S' \not\sim S_i'$ for some splices S_i'. Since Φ^* is a bijection, $\exists$ splices S_i s.t. $\Phi(S_i) \sim S_i'$. So

$$\Phi({}^{\triangleleft}F) \sim S_1' \cup \dots \cup S_n' \sim \Phi(S_1) \cup \dots \cup \Phi(S_n)$$

$$\Rightarrow \Phi^{-1}(\Phi({}^{\triangleleft}F)) \sim \Phi^{-1}(\Phi(S_1) \cup \dots \cup \Phi(S_n))$$

$$= \Phi^{-1}(\Phi(S_1)) \cup \dots \cup \Phi^{-1}(\Phi(S_n))$$

$$\Rightarrow {}^{\triangleleft}F \sim S_1 \cup \dots \cup S_n$$

$$\Rightarrow \exists i : {}^{\triangleleft}F \sim S_i \Rightarrow \Phi({}^{\triangleleft}F) \sim \Phi(S_i)$$

$\Rightarrow S' \sim S_i'$ which is a contradiction. ∎

- Hence Φ^* sends Weyl chambers and chamber walls, to Weyl chambers and chamber walls.

- Since Weyl chambers of the max. dim., we get that $\Phi^*(\xi)$ consists of Weyl chambers. So we get $\Phi_0 : (X_1)_0 \longrightarrow (X_2)_0$. Since Φ^* is Γ-equi. and bijection, we get that Φ_0 is Γ-equi. and bijection. ∎

Lecture 21: Map on the set of parabolics

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In fact the above argument gives us a bit more:

For any Weyl chamber or chamber wall ${}^{\triangleleft}S$, let $[{}^{\triangleleft}S]$ be its class in $\mathcal{S}(X)$. Then all of elements of $[{}^{\triangleleft}S]$ are again either a Weyl chamber or a chamber wall. Let $\mathcal{S}^{irr}(X)$ be the set of such elements. Then Φ^* induces a Γ-equiv. bijection $\Phi^*_{irr} : \mathcal{S}^{irr}(X_1) \longrightarrow \mathcal{S}^{irr}(X_2)$.

On the other hand, to any ${}^{\triangleleft}S$, we attached a parabolic $P({}^{\triangleleft}S)$. And one can see (similar to the discussion that we had for maximal boundary)

$${}^{\triangleleft}S \sim {}^{\triangleleft}S' \iff {}^{\triangleleft}S' = g\,{}^{\triangleleft}S \text{ for some } g \in P({}^{\triangleleft}S)$$

And from here one can deduce $P({}^{\triangleleft}S) = P({}^{\triangleleft}S')$ and

$$[{}^{\triangleleft}S] \underset{P_X}{\longmapsto} P({}^{\triangleleft}S) \quad \text{is a bijection between}$$

$$\mathcal{S}^{irr}(X) \longrightarrow T(G) := \text{the set of parabolics.}$$

<u>Corollary</u>. $\exists\ \tilde{\Phi} : T(G_1) \longrightarrow T(G_2)$ a bijection s.t.

$$\forall\ \gamma \in \Gamma, \quad \tilde{\Phi}(\gamma P \gamma^{-1}) = \theta(\gamma)\, \tilde{\Phi}(P)\, \theta(\gamma)^{-1}.$$

Lecture 21: Isometry of Tits spherical buildings

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To get an isomorphism between G_i's, using Tits's result, it is enough to show that the mentioned bijection $\tilde{\Phi}: T(G_1) \to T(G_2)$ preserves the ordering.

$$\mathcal{P}({}^{\triangleleft}S_1) \subseteq \mathcal{P}({}^{\triangleleft}S_2) \Rightarrow {}^{\triangleleft}S_2 \subseteq N_{O(1)}({}^{\triangleleft}S_1)$$

$$\Rightarrow \Phi({}^{\triangleleft}S_2) \subseteq N_{O(1)}(\Phi({}^{\triangleleft}S_1))$$

$$\Rightarrow \Phi^*_{irr}({}^{\triangleleft}S_2) \subseteq N_{O(1)}(\Phi^*_{irr}({}^{\triangleleft}S_1))$$

$$\Rightarrow \mathcal{P}(\Phi^*_{irr}({}^{\triangleleft}S_1)) \subseteq \mathcal{P}(\Phi^*_{irr}({}^{\triangleleft}S_2))$$

$$\Rightarrow \tilde{\Phi}(\mathcal{P}({}^{\triangleleft}S_1)) \subseteq \tilde{\Phi}(\mathcal{P}({}^{\triangleleft}S_2)). \blacksquare$$

So by the rigidity of Tits spherical building we get

$\exists\ \tilde{\theta}: G_1 \to G_2$ s.t. $\tilde{\Phi}(\mathcal{P}) = \tilde{\theta}(\mathcal{P})$.

$$\text{So} \quad \tilde{\theta}(\gamma \mathcal{P} \gamma^{-1}) = \theta(\gamma)\, \tilde{\theta}(\mathcal{P})\, \theta(\gamma)^{-1}$$

$$\Rightarrow \theta(\gamma)\, \tilde{\theta}(\gamma)^{-1} \in \bigcap_{P \in T(G_2)} N(P) = \bigcap_{P \in T(G_2)} P = 1$$

(as $Z(G_2)=1$ and it has no compact factors.)

$$\Rightarrow \theta(\gamma) = \tilde{\theta}(\gamma) \Rightarrow \tilde{\theta}|_{\Gamma} = \theta. \blacksquare$$