Lecture7: Haar measure and volume forms for SL(2,R)

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<u>Theorem</u>. G: second countable, Hausdorff, locally compact,

$\exists !$ (up to a scalar) regular measure λ_G which is left G-invariant.
It is called a left Haar measure.

- $\forall g \in G$, $\underline{\lambda_G \cdot g}$ is another left Haar measure $\Rightarrow \exists \Delta_G : G \to \mathbb{R}^+$ s.t.
  - $\lambda_G \cdot g = \Delta_G(g)\, \lambda_G \Rightarrow \Delta_G$ is a group homomorphism
  - regularity $\Rightarrow \Delta_G$ is continuous.

  So Δ_G factors through $G/\overline{[G,G]}$.

<u>Corollary</u>. • G: semisimple Lie group $\Rightarrow$ G is unimodular, i.e. $\Delta_G = 1$, i.e. a left Haar measure is also a right Haar measure.

- G compact $\Rightarrow$ G is unimodular as $\mathbb{R}^+$ has no non-trivial compact subgp.
- G discrete $\Rightarrow$ counting measure is a left and right Haar measure.
- (G abelian $\Rightarrow$ unimodular).

- <u>Another measure on SL_2</u>.

How can we use the manifold structure?

<u>Proposition</u>. $G \subseteq \mathbb{R}^n$ an open subset;

G is a topological group

$$\Rightarrow \quad \frac{d\vec{x}}{|\det(dl_x|_e)|} \quad \text{is } G\text{-invariant},$$

where $l_x : G \to G$, $l_x(x') = x \cdot x'$.

Pf. Suppose $f(\vec{x})\, d\vec{x}$ is G-invariant. Then

$$f(l_{\vec{x}'}(\vec{x}))\; d(l_{\vec{x}'}(x)) = f(\vec{x})\; d\vec{x}$$

$$\parallel$$

$$f(l_{\vec{x}'}(\vec{x}))\; |\det(dl_{\vec{x}'}|_{\vec{x}})|\; d\vec{x}$$

For $\vec{x} = \vec{e} \Rightarrow$

$$f(\vec{x}') = \frac{f(\vec{e})}{|\det(dl_{x'}|_e)|}.$$

This is how one can come up with the above proposition.

We need to check the following:

$$\frac{1}{|\det(dl_{(l_{\vec{x}'}(\vec{x}))}|_e)|}\; |\det(dl_{\vec{x}'}|_{\vec{x}})| \overset{?}{=} \frac{1}{|\det(dl_{\vec{x}}|_e)|}.$$

$$l_{(l_{x'}(x))} = l_{x'} \circ l_x \Rightarrow dl_{(l_{x'}(x))}\Big|_e = dl_{x'}\Big|_x \circ dl_x\Big|_e$$

$$\Rightarrow \det(dl_{(l_{x'}(x))}|_e) = \det(dl_{x'}|_x) \cdot \det(dl_x|_e).$$

∎

Corollary. $\dfrac{dx_{11}\, dx_{12} \cdots dx_{nn}}{(\det X)^n}$ is GL_n-invariant.

Corollary: $\dfrac{\cdots}{(\det X)^n}$ is GL_n-invariant.

Pf. $\left.\begin{array}{l} \ell_X(y) = Xy \\ \text{identify } GL_n \subseteq \mathbb{A}^{n^2} \end{array}\right\} \Rightarrow d\ell_X\big|_e = \begin{bmatrix} X & & \\ & \ddots & \\ & & X \end{bmatrix}$. ∎

Corollary.. $\dfrac{dy\,dx}{y^2}$ is left B-invariant where $B = \left\{ \begin{bmatrix} y & x \\ & y^{-1} \end{bmatrix} \,\middle|\, \begin{array}{l} y \in \mathbb{G}_m, \\ x \in \mathbb{G}_a \end{array} \right\}$

- $dy\,dx$ is right B-invariant.

Pf. $B \hookrightarrow \mathbb{A}^2$

$\begin{bmatrix} y & x \\ & y^{-1} \end{bmatrix} \longmapsto (y, x)$

$\ell_{(y,x)}(a,b) = (ya, yb + xa^{-1}) \Rightarrow d\ell_{(y,x)}\big|_{(1,0)} = \begin{bmatrix} y & -x \\ & y \end{bmatrix}$

$\begin{bmatrix} y & x \\ & y^{-1} \end{bmatrix}\begin{bmatrix} a & b \\ & a^{-1} \end{bmatrix} = \begin{bmatrix} ya & yb + xa^{-1} \\ & (ya)^{-1} \end{bmatrix} \Rightarrow \dfrac{dy\,dx}{y^2}$ is left B-invar.

$r_{(y,x)}(a,b) = (ya, xa + y^{-1}b) \Rightarrow dr_{(y,x)}\big|_{(1,0)} = \begin{bmatrix} y & x \\ & y^{-1} \end{bmatrix}$

$\begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix}\begin{bmatrix} y & x \\ 0 & y^{-1} \end{bmatrix} = \begin{bmatrix} ay & ax + by^{-1} \\ 0 & a^{-1}y^{-1} \end{bmatrix} \Rightarrow dy\,dx$ is right B-invar. ∎

What if we change our "local coordinates"?

$B = \{ n(x)a(y) \mid x \in \text{add}, y \in \text{mult.} \}$

$\ell_{(x,y)}(x', y')$ $(n(x)a(y))(n(x')a(y')) = n(x)\, n(y^2x')\, a(yy') = n(x + y^2x')\, a(yy')$

$\| $

$(x + y^2x', yy') \Rightarrow d\ell_{(x,y)}\big|_{(0,1)} = \begin{bmatrix} y^2 & 0 \\ 0 & y \end{bmatrix} \Rightarrow \boxed{\dfrac{1}{y^2} \cdot \dfrac{dy}{y} \cdot dx}$

Theorem. $\left.\begin{array}{ll} G_1, G_2 \subseteq G & \text{closed subgroups} \\ G_1G_2 \subseteq G & \text{open subgroup} \\ G_1 \cap G_2 & \text{compact} \end{array}\right\} \Rightarrow$ λ_G can be defined as follows (up to a scalar)

$G_1 G_2 = G$ open subgroup
$G_1 \cap G_2$ compact } as follows (up to a scalar)

$$\int_{G_1}\int_{G_2} f(g_1 g_2)\, \frac{\Delta_{G_2}(g_2)}{\Delta_G(g_2)}\, d\lambda_{G_2}(g_2)\, d\lambda_{G_1}(g_1) =: \int_G f(g)\, d\lambda_G(g)$$

<u>Corollary</u> $\quad f(g) \overset{\mu_2}{\longmapsto} \displaystyle\int_{\mathbb{R}} \int_{\mathbb{R}^+} \int_K f(n(x)\, a(y)\, k)\, \frac{1}{y^2}\, dk\, \frac{dy}{y}\, dx$

defines a (left) Haar measure of $SL_2(\mathbb{R})$ where $K = SO_2(\mathbb{R})$.

<u>Pf.</u> $SL_2(\mathbb{R}) = B \cdot K \quad \Longrightarrow \quad \checkmark$

$B \cap K$ is finite

$\lambda_B = \frac{1}{y^2} \cdot \frac{dy}{y} \cdot dx$

$\Delta_K = 1$, $\Delta_{SL_2} = 1$: they are unimodular. ∎

<u>Comapre these two measures on $SL_2(\mathbb{R})$.</u>

$f \in C_c(SL_2(\mathbb{R})) \mapsto \bar{f}(x+iy) := \displaystyle\int_{SO_2(\mathbb{R})} f(n(x)\, a(\sqrt{y})\, k)\, dk$

where $n(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$ and $a(y) = \begin{bmatrix} y & \\ & y^{-1} \end{bmatrix}$,

and $f \longmapsto \mu_1(f) := \displaystyle\int_{\mathcal{H}} \bar{f}(z)\, d_{\mathcal{H}} z$.

<u>Why is it a left Haar measure?</u>

$\forall g \in SL_2(\mathbb{R})$, $L_g f : SL_2(\mathbb{R}) \to \mathbb{R}$,

$(L_g f)(g') := f(g^{-1} g')$.

$\Rightarrow \mu_1(L_g f) = \displaystyle\int_{\mathcal{H}} \overline{L_g f}\,(z)\, d_{\mathcal{H}}(z)$

$$\overline{L_g f}(z) = \int_K f(g^{-1} n(x) a(\sqrt{y})\, k)\, dk$$

$$= \int_K f\left(n(\mathrm{Re}(g^{-1}\cdot z))\, a\left(\sqrt{\mathrm{Im}(g^{-1}\cdot z)}\right) \sigma_K(g^{-1}, z)\, k\right) dk$$

where $\sigma_K : G \times \mathcal{H} \longrightarrow K$ is a cocycle

$$g\, n(\mathrm{Re}(z))\, a(\mathrm{Im}(z)) = n(\mathrm{Re}(g.z))\, a(\mathrm{Im}(g.z))\, \sigma_K(g, z)$$

$$= \int_K f(n(\mathrm{Re}(g^{-1}\cdot z))\, a(\sqrt{\mathrm{Im}(g^{-1}\cdot z)})\, k)\, dk$$

$$= \overline{f}(g^{-1}\cdot z)$$

$$= (L_g \overline{f})(z).$$

$$\Rightarrow \mu_1(L_g(\overline{f})) = \int_{\mathcal{H}} L_g(\overline{f})(z)\, d_{\mathcal{H}} z = \int_{\mathcal{H}} \overline{f}(z)\, d_{\mathcal{H}} z = \mu_1(f). \quad \blacksquare$$

$$\mu_1(f) = \int_{\mathbb{R}} \int_{\mathbb{R}^+} \overline{f}(x+iy) \frac{dy}{y^2}\, dx$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}^+} \int_K f(n(x)\, a(\sqrt{y})\, k) \cdot \frac{1}{y^2}\, dk\, dy\, dx$$

$$y' = \sqrt{y} \Rightarrow 2y'\, dy' = dy \Rightarrow \frac{dy}{y^2} = \frac{2y'\, dy'}{y'^4} = \frac{2}{y'^2} \cdot \frac{dy'}{y'}.$$

$$= 2 \int_{\mathbb{R}} \int_{\mathbb{R}^+} \int_K f(n(x)\, a(y')\, k) \frac{1}{y'^2}\, dk\, \frac{dy'}{y'}\, dx$$

$$\Rightarrow \mu_1 = 2\mu_2$$

$$\Rightarrow \mu_2\left({}_{SL_2(\mathbb{Z})}\backslash^{SL_2(\mathbb{R})}\right) = \frac{\pi^2}{6} = \zeta(2).$$

Third way of getting a volume form on $SL_2(\mathbb{R})$.

$$\left\{\begin{bmatrix} t & y \\ z & x \end{bmatrix} \;\middle|\; xt - yz = 1\right\}$$

Take x, y, and z as local coordinates. We would like to find f s.t. $f(x,y,z)\, dx \wedge dy \wedge dz$ is left-SL_2-invariant:

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \theta \end{bmatrix}\begin{bmatrix} t & y \\ z & x \end{bmatrix} = \begin{bmatrix} \alpha t + \beta z & \alpha y + \beta x \\ \gamma t + \theta z & \gamma y + \theta x \end{bmatrix}$$

$$d(\gamma y + \theta x) \wedge d(\alpha y + \beta x) \wedge d(\gamma t + \theta z)$$

$$= dx \wedge dy \wedge (\gamma\, dt + \theta\, dz)$$

$$xt - yz = 1 \Rightarrow x\, dt + t\, dx = y\, dz + z\, dy$$

$$\Rightarrow x\, dx \wedge dy \wedge dt = y\, dx \wedge dy \wedge dz$$

$$= \left(\theta + \gamma \frac{y}{x}\right) dx \wedge dy \wedge dz = \frac{\theta x + \gamma y}{x}\, dx \wedge dy \wedge dz.$$

$$\Rightarrow \quad f(g \cdot \underline{g}(x,y,z))\; x_{22}(g \cdot \underline{g}) = f(\underline{g})\; x_{22}(\underline{g})$$

$$\Rightarrow \quad f(g) = \frac{\text{Const.}}{x_{22}(g)}.$$

$\Rightarrow \quad \frac{1}{x} dx \wedge dy \wedge dz$ is a left SL_2-invar.

Comparing measures.

$$\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} y & \\ & y^{-1} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} y & xy^{-1} \\ & y^{-1} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} y\cos\theta - xy^{-1}\sin\theta & y\sin\theta + xy^{-1}\cos\theta \\ -y^{-1}\sin\theta & y^{-1}\cos\theta \end{bmatrix}$$

$$d(y^{-1}\cos\theta) = \frac{-1}{y^2}\cos\theta \, dy - \frac{\sin\theta}{y} d\theta$$

$$d(y\sin\theta + xy^{-1}\cos\theta) = y^{-1}\cos\theta \, dx + *$$

$$d(-y^{-1}\sin\theta) = \frac{1}{y^2}\sin\theta \, dy - \frac{\cos\theta}{y} d\theta$$

$$\Rightarrow d(y^{-1}\cos\theta) \wedge d(y\sin\theta + xy^{-1}\cos\theta) \wedge d(-y^{-1}\sin\theta)$$

$$= -y^{-1}\cos\theta \, dx \wedge d(y^{-1}\cos\theta) \wedge d(-y^{-1}\sin\theta)$$

$$= -y^{-1}\cos\theta \left(\frac{1}{y^3}\cos^2\theta + \frac{1}{y^3}\sin^2\theta\right) dx \wedge dy \wedge d\theta$$

$$= -\frac{1}{y^4}\cos\theta \, dx \wedge dy \wedge d\theta$$

$$\rightsquigarrow \frac{1}{x} dx \wedge dy \wedge dz = \frac{1}{y^{-1}\cos\theta} \cdot -y^4\cos\theta \cdot dx \wedge dy \wedge d\theta$$

(old coordinates)
I used the same
variables by mistake

$$= \frac{-1}{y^3} dx \wedge dy \wedge d\theta \, .$$

So the induced measure is the same as μ_2.

What is the advantage of $\frac{1}{x}\, dx \wedge dy \wedge dz$?

This is an algebraic form so it induces Haar measure over other local fields.

Ex. Find $|\omega|_p(SL_2(\mathbb{Z}_p))$.

Solution. $|\omega|_p(SL_2(\mathbb{Z}_p)) = |SL_2(\mathbb{F}_p)|\ |\omega|_p(SL_2^1(\mathbb{Z}_p))$

where $SL_2^1(\mathbb{Z}_p) = \{ I + pX \mid \det(I+pX) = 1 \}$.

$$(1+px', py', pz') \longmapsto \begin{bmatrix} \frac{1+(py')(pz')}{1+px'} & (py') \\ (pz') & 1+px' \end{bmatrix} \in SL_2^1(\mathbb{Z}_p)$$

is a well-defined

p-adic analy. map.

To show this it is enough to note that

$$\frac{1+(py')(pz')}{1+px'} - 1 = \frac{p^2y'z' - px'}{1+px'} \in p\mathbb{Z}_p.$$

So $|\omega|_p(SL_2^1(\mathbb{Z}_p)) = \displaystyle\int_{p\mathbb{Z}_p^3} \frac{1}{|1+x|_p}\, dx\, dy\, dz$

$$= \mathrm{vol}(p\mathbb{Z}_p^3) = \frac{1}{p^3}.$$

$$\Rightarrow \quad |\omega|_p(SL_2(\mathbb{Z}_p)) = \frac{|SL_2(\mathbb{F}_p)|}{p^3} = \frac{(p^2-1)(p^2-p)}{(p-1)\ p^3}$$

$$= 1 - p^{-2}. \quad \blacksquare$$

Observation $|\omega|_\infty\left(SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})\right) \cdot \prod |\omega|_p(SL_2(\mathbb{Z}_p))$

Observation $\quad |\omega|_\infty\left(SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})\right)\cdot\prod_p |\omega|_p(SL_2(\mathbb{Z}_p))$

$$= \zeta(2)\cdot\prod_p(1-p^{-2}) = 1.$$

Strong approximation $\quad SL_2(\mathbb{A}) = \underbrace{\Delta(SL_2(\mathbb{Q}))}_{\text{diag. embed.}}\cdot\left(SL_2(\mathbb{R})\cdot\prod_p SL_2(\mathbb{Z}_p)\right)$

$\boxed{\text{Tamagawa Number of } SL_2/\mathbb{Q}}$

$$SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A}) \longleftrightarrow SL_n(\mathbb{Z})\backslash SL_2(\mathbb{R}) \times \prod_p SL_2(\mathbb{Z}_p)$$

$$|\omega|\left(SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A})\right) = |\omega|_\infty\left(SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})\right)\cdot\prod_p|\omega|_p(SL_2(\mathbb{Z}_p))$$

$$= 1.$$