Monday, October 19, 2015 2:01 PM

<u>Proposition</u> $\Theta : GL_n(\mathbb{R})/GL_n(\mathbb{Z}) \to \Omega(\mathbb{R}^n)$, $\Theta(g\,GL_n(\mathbb{Z})) := g\mathbb{Z}^n$ is a homeomorphism.

<u>Pf.</u>

We have already proved that Θ is a $GL_n(\mathbb{R})$-equivariant bijection. So it is enough to show it is conti. and open around identity.

- <u>Θ is continuous</u>.

$$g_m GL_n(\mathbb{Z}) \xrightarrow[m\to\infty]{} GL_n(\mathbb{Z}) \Rightarrow \exists\, \gamma_m \in GL_n(\mathbb{Z}) \text{ s.t. } g_m\gamma_m \to I. \quad (*)$$

$$g_m\mathbb{Z}^n = \overbrace{g_m\gamma_m}^{g'_m}\mathbb{Z}^n = \bigoplus_{i=1}^{n} \mathbb{Z}\, v_m^{(i)} \quad \text{where}$$

$g_m\gamma_m = [v_m^{(1)} \cdots v_m^{(n)}]$. So by $(*)$ we have

$$v_m^{(i)} \xrightarrow[m\to\infty]{} e_i$$

- $B_0(R) \cap g'_m \mathbb{Z}^n = g'_m\left(g'^{-1}_m B(R) \cap \mathbb{Z}^n\right)$

Since $g'_m \to I$, $\forall \varepsilon > 0$ and $m \gg_{\varepsilon, R} 1$ we have

$$g'^{\pm 1}_m B(R) \subseteq B(2R)$$

and $\forall\, v \in B(2R)$, $\| g'^{\pm 1}_m v - v \| \le \varepsilon$.

$$\Rightarrow d_{\text{Haus}}\left(g_m\mathbb{Z}^n \cap B(R),\ \mathbb{Z}^n \cap B(R)\right) \le \varepsilon.$$

- <u>Θ is open.</u>

Suppose $g_m\mathbb{Z}^n \to \mathbb{Z}^n$, i.e., for any $R > 0$ and $\varepsilon > 0$, Suppose $g_m \mathbb{Z} \longrightarrow \mathbb{Z}$, i.e., for any $R>0$ and $\varepsilon>0$,

if $m \gg_{R,\varepsilon} 1$, then

I $\quad \operatorname{dist}_{\text{Haus}}\left(g_m \mathbb{Z}^n \cap (-R,R)^n \,,\ \mathbb{Z}^n \cap (-R,R)^n\right) \leq \varepsilon . \Rightarrow$ if $m \gg_{R,\varepsilon} 1$,

$$\forall\, v \in \mathbb{Z}^n \cap (-R,R)^n ,\ \exists!\, w \in \mathbb{Z}^n \cap g_m^{-1}(-R,R)^n$$

$$\| g_m w - v \| \leq \varepsilon .$$

(Existence is clear; uniqueness?

If $\omega_1 \neq \omega_2$ satisfy these properties, then

$$g_m \omega_1 - g_m \omega_2 = g_m(\omega_1 - \omega_2) \in g_m \mathbb{Z}^n \cap B(2\varepsilon, \vec{0})$$

$\Longrightarrow$ For some $k \in \mathbb{Z}^{>0}$,

$$k\, g_m(\omega_1-\omega_2) \in g_m \mathbb{Z}^n \cap \left(B(1/2+\varepsilon, 0) \setminus B(\tfrac{1}{2}-\varepsilon, 0)\right)$$

$\Longrightarrow \ \exists\, v' \in \mathbb{Z}^n \cap (-R,R)^n$ s.t.

$$\| k g_m(\omega_1 - \omega_2) - v' \| \leq \varepsilon \qquad \text{II}$$

$$\Longrightarrow \qquad \| v' \| \leq \tfrac{1}{2} + 2\varepsilon .$$

$\Longrightarrow \quad v' = \vec{0} \ \overset{\text{II}}{\Longrightarrow}\ \frac{1}{2} - \varepsilon \leq \varepsilon \quad$ which is a contradiction. ∎)

Let $f_{R,\varepsilon,m}(\vec{v}) := \vec{w}$ for $m \gg_{R,\varepsilon} 1$.

Let $\gamma_m^{(\varepsilon)} := \left[f_{2,\varepsilon,m}(e_1) \cdots f_{2,\varepsilon,m}(e_n) \right] \in M_n(\mathbb{Z})$ for $m \gg_\varepsilon 1$.

Notice that $\gamma_m^{(\varepsilon_1)} = \gamma_m^{(\varepsilon_2)}$ if $m \geq g(\varepsilon)$ (by uniqueness.)

So $\| g \ \gamma^{(\varepsilon)} \ T \quad \| < \varepsilon$ for any $m > g(\varepsilon)$

$$\boxed{\text{So } \left\| g_m \gamma_m^{(\varepsilon)} - I \right\| \le \varepsilon \text{ for any } m \ge g(\varepsilon)}$$

Let $\gamma_m = \gamma_m^{(2^{-k})}$ if $g(2^{-k}) \le m < g(2^{-k-1})$

(if $\lim_{\varepsilon \to 0} g(\varepsilon) \ne \infty$, then let $\gamma_m = \gamma_m^{(0)}$ for $m \ge \lim_{\varepsilon \to 0} g(\varepsilon)$.)

Hence $\| g_m \gamma_m - I \| \le 2^{-k}$ if $m \ge g(2^{-k})$

$$\Rightarrow \boxed{g_m \gamma_m \longrightarrow I \text{ as } m \to \infty.}$$

<u>Claim</u> $\gamma_m \in GL_n(\mathbb{Z})$ for $m \gg 1$.

<u>Pf of claim</u>. It is enough to show $\gamma_m^{-1} \mathbb{Z}^n = \mathbb{Z}^n$.

Suppose $\gamma_m (\sum c_i^{(m)} \vec{e}_i) \in \mathbb{Z}^n$. W.L.O.G we can and will assume $|c_i^{(m)}| \le \frac{1}{2}$. Suppose to the contrary that $\frac{1}{4} \le \max |c_i^{(m)}| \le \frac{1}{2}$

We can assume $\max |c_i^{(m)}| = |c_1^{(m)}|$ (passing to a subseq.& rearr.).

- $g_m \gamma_m (\sum c_i^{(m)} \vec{e}_i) \in g_m \mathbb{Z}^n$
- passing to a subseq. if needed, $\sum c_i^{(m)} \vec{e}_i \longrightarrow \sum c_i \vec{e}_i$

$$\Rightarrow g_m \gamma_m (\sum c_i^{(m)} \vec{e}_i) \xrightarrow[m \to \infty]{} \sum c_i \vec{e}_i \quad \circledast$$

Take an ε-nbhd $\mathcal{O}$ of $\underline{\sum_{m \to \infty} c_i \vec{e}_i}$. By $\circledast$, $g_m \mathbb{Z}^n \cap \mathcal{O} \ne \emptyset$ for $m \gg_{\mathcal{O}} 1$. Since $g_m \mathbb{Z}^n \longrightarrow \mathbb{Z}^n$, we conclude that

$\mathbb{Z}^n \cap \mathcal{O} \ne \emptyset$ which is a contradiction as

$$\frac{1}{4} \le \|\vec{c}\|_\infty \le 1/2. \quad \blacksquare$$

$$\frac{1}{4} \leq \|\vec{c}\|_\infty \leq 1/2 . \quad \blacksquare$$

Corollary. $SL_n(\mathbb{R})/SL_n(\mathbb{Z}) \xrightarrow{\sim} \Omega^{(1)}(\mathbb{R}^n) := \{\Delta \in \Omega(\mathbb{R}^n) \mid \mathrm{vol}(\mathbb{R}^n/\Delta) = 1\}.$

Pf. • Ⓗ induces a bijection * $g \in SL_n(\mathbb{R}) \Rightarrow \mathrm{vol}(\mathbb{R}^n/g\mathbb{Z}^n) = |\det(g)| = 1.$

* $\mathrm{vol}(\mathbb{R}^n/g\mathbb{Z}^n) = 1 \Rightarrow |\det(g)| = 1.$

$$g\mathbb{Z}^n = g\omega\,\mathbb{Z}^n$$

where $\omega = \begin{bmatrix} \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & \\ & I \end{bmatrix}$ and $\det(g\omega) = -\det(g)$.

• Since Ⓗ was a homeomorphism, we are done. $\blacksquare$

• Lemma. $\delta : \Omega(\mathbb{R}^n) \to \mathbb{R}^+$, $\delta(\Delta) := \min\{\|v\| \mid v \in \Delta \setminus \{0\}\}$

is continuous.

Pf. $\Delta_m \xrightarrow[m\to\infty]{} \Delta$.

• Let $v \in \Delta \setminus \{0\}$ s.t. $\|v\| = \delta(\Delta)$. Then, for any $\varepsilon > 0$

and $m \gg_\varepsilon 1$, $\Delta_m \cap B(\varepsilon; v) \neq \emptyset$.

$\Rightarrow \delta(\Delta_m) \leq \delta(\Delta) + \varepsilon$

$\Rightarrow \overline{\lim_{m\to\infty}}\, \delta(\Delta_m) \leq \delta(\Delta)$.

• Let $\underline{\lim}_{m\to\infty}\, \delta(\Delta_m) = \delta_0$. Passing to a subsequence $\exists\, v_m \in \Delta_m$

$m \to \infty$

that $v_m \to v'$ and $\|v'\| = \delta_0$.

. If $\delta_0 \neq 0$, then

$$\left.\begin{array}{c} \Delta_m \longrightarrow \Delta \\ \rotatebox{90}{$\in$} \\ v_m \longrightarrow v' \end{array}\right\} \Rightarrow v' \in \Delta \setminus \{0\} \Rightarrow \delta_0 \geq \delta(\Delta)$$

and we are done.

. If $\delta_0 = 0$, then $\|v_m\| \to 0$ & $\|v_m\| \neq 0$

$\Rightarrow$ after multiplying by a suitable integer k_m we have $\delta(\Delta)/_4 \leq \| k_m v_m \| \leq \delta(\Delta)/_2$

Passing to a subseq.,

$$\left.\begin{array}{c} k_m v_m \longrightarrow v'' \\ \rotatebox{-90}{$\in$} \\ \Delta_m \longrightarrow \Delta \end{array}\right\} \Rightarrow \left.\begin{array}{l} v'' \in \Delta \\ \delta(\Delta)/_4 \leq \|v''\| \leq \delta(\Delta)/_2 \end{array}\right\}$$

which is a contradiction. ∎

<u>Corollary</u>. If $X \subseteq \Omega(\mathbb{R}^n)$ is precompact, then

$\exists\, \delta_0 > 0$: $\delta_0 \leq \delta(X) \leq 1/\delta_0$