In the previous lectures we studied the following algorithm:

Input $\Delta \in \Omega(\mathbb{R}^n)$

Output $v_1, \dots, v_n \in \Delta$

Aux. $V_0 = 0$; $\quad \Delta_0 = 0$;

$V_k = \bigoplus_{i=1}^{k} \mathbb{R} v_i$; $\quad \Delta_k = \bigoplus_{i=1}^{k} \mathbb{Z} v_i$;

For $k = 0 \dots n-1$

1. Choose $v_{k+1} \in \Delta$ s.t. $\|\mathrm{Pr}_{V_k^\perp}(v_{k+1})\| = \delta(\mathrm{Pr}_{V_k^\perp}(\Delta))$;
2. $V_{k+1} := V_k \oplus \mathbb{R} v_{k+1}$;
3. $\Delta_{k+1} := \Delta_k \oplus \mathbb{Z} v_{k+1}$;

We proved that $\Delta_k = \Delta \cap V_k$.

Let $\omega_k := \mathrm{Pr}_{V_{k-1}^\perp}(v_k)$ for $1 \le k \le n$. This is what Gram–Schmidt process gives us for an orthogonal basis for V_k.

Lemma 1 $V_k = \Delta_k + \mathcal{F}_k$ where $\mathcal{F}_k := \{ \sum_{i=1}^{k} c_i \omega_i \mid |c_i| \le 1/2 \}$

Pf. We use induction on k. For $k=0$, it is clear.

$v \in V_k \Rightarrow v = c\, v_k + v'_{k-1}$ for some $c \in \mathbb{R}$, $v'_{k-1} \in V_{k-1}$.

$$\Rightarrow \mathrm{Pr}_{V_{k-1}^\perp}(v) = c\, \omega_k = (\ell + \alpha_k)\, \omega_k$$

where $\ell \in \mathbb{Z}$ and $|\alpha_k| \le 1/2$

where $l \in \mathbb{Z}$ and $|\alpha_k| \le 1/2$

$$\Rightarrow v - l v_k = Pr_{V_{k-1}^{\perp}}(v - l v_k) + Pr_{V_{k-1}}(v - l v_k)$$

$$\in \alpha_k w_k + \Delta_{k-1} + \mathcal{F}_{k-1}$$

by the induction hypothesis

$$\Rightarrow v \in \mathbb{Z} v_k + \Delta_{k-1} + \alpha_k w_k + \mathcal{F}_{k-1} \subseteq \Delta_k + \mathcal{F}_k . \quad \blacksquare$$

<u>Lemma 2</u> $\{w_1, \dots, w_k\}$ is an othogonal basis of V_k.

<u>Proof.</u> Inductively it is clear as $w_{k+1} = Pr_{V_k^{\perp}}(v_{k+1})$. $\blacksquare$

<u>Cor. 3</u> For $v \in V_k$, $v \in \mathcal{F}_k \Leftrightarrow \left| \frac{v \cdot w_i}{w_i \cdot w_i} \right| \le 1/2$ for $1 \le i \le k$.

<u>Pf</u>. $v = \sum_{i=1}^{k} c_i w_i \Rightarrow v \cdot w_j = c_j \, w_j \cdot w_j$

$$\Rightarrow v = \sum_{j=1}^{k} \frac{v \cdot w_j}{w_j \cdot w_j} w_j . \quad \blacksquare$$

Now in the above algorithm among all possible v_{k+1}'s in <u>step 1</u>.
Choose the one such that $Pr_{V_k}(v_{k+1}) \in \mathcal{F}_k$. We can do this by $Pr_{V_k^{\perp}}(\Delta_k) = 0$ and Lemma 1.

- Since $Pr_{V_{k-1}}(v_k) \in \mathcal{F}_{k-1}$, by Corollary 3 for $1 \le j \le k$

$$\frac{1}{2} \ge \left| \frac{Pr_{V_{k-1}}(v_k) \cdot w_j}{w_j \cdot w_j} \right| = \left| \frac{v_k \cdot w_j}{w_j \cdot w_j} \right| .$$

$$| \quad w_j \quad w_j \; | \qquad | \; w_j \quad w_j \; |$$

- Since $\left\| Pr_{V_{k-1}^{\perp}}(v_k) \right\| = \delta\left(Pr_{V_{k-1}^{\perp}}(\Delta) \right)$,

$$\|w_k\| = \left\| Pr_{V_{k-1}^{\perp}}(v_k) \right\| \le \left\| Pr_{V_{k-1}^{\perp}}(v_{k+1}) \right\|$$

$$= \left\| \left(\frac{v_{k+1} \cdot w_k}{w_k \cdot w_k} \right) w_k + w_{k+1} \right\|$$

$$\Rightarrow \|w_k\|^2 \le \left| \frac{v_{k+1} \cdot w_k}{w_k \cdot w_k} \right|^2 \|w_k\|^2 + \|w_{k+1}\|^2$$

$$\le \frac{1}{4} \|w_k\|^2 + \|w_{k+1}\|^2$$

$$\Rightarrow \quad \frac{\|w_k\|}{\|w_{k+1}\|} \le \frac{2}{\sqrt{3}} .$$

- Let $A_\alpha := \left\{ \operatorname{diag}(a_1, \dots, a_n) \;\middle|\; a_k / a_{k+1} \le \alpha ,\ a_k \in \mathbb{R}^+ \right\}$,

$A_\alpha^{(1)} := A_\alpha \cap SL_n(\mathbb{R})$,

$N_\beta := \left\{ \begin{bmatrix} 1 & n_{ij} \\ & \ddots \\ & & 1 \end{bmatrix} \;\middle|\; |n_{ij}| \le \beta \right\}$,

$$\left. \begin{array}{l} \Sigma_{\alpha,\beta} := K' A_\alpha N_\beta \quad \text{where} \quad K' = O(n), \\ \Sigma_{\alpha,\beta}^{(1)} := K A_\alpha^{(1)} N_\beta \quad \text{where} \quad K = SO(n). \end{array} \right\} \text{Siegel sets.}$$

<u>Theorem</u> ① $GL_n(\mathbb{R}) = \Sigma_{2/\sqrt{3}, 1/2} \cdot GL_n(\mathbb{Z})$.

② $SL_n(\mathbb{R}) = \Sigma_{2/\sqrt{3}, 1/2}^{(1)} \cdot SL_n(\mathbb{Z})$.

<u>Proof.</u> ① $\forall\, g \in GL_n(\mathbb{R})$, $g\mathbb{Z}^n$ has a basis $v_1, \dots, v_n$ with the above

n

properties $\Rightarrow g\mathbb{Z}^n = [v_1 \cdots v_n]\mathbb{Z}^n$

$$\Rightarrow \exists\, \gamma \in GL_n(\mathbb{Z}) \text{ s.t. } g\gamma^{-1} = [v_1 \cdots v_n].$$

By Gram-Schmidt process, for any $1 \le k \le n$,

$$[v_1 \cdots v_k] = [w_1 \cdots w_k]\begin{bmatrix} 1 & & \frac{v_i\cdot w_j}{w_j\cdot w_j} \\ & \ddots & \\ & & 1 \end{bmatrix}$$

as $v_k = \dfrac{v_k\cdot w_1}{w_1\cdot w_1} w_1 + \cdots + \dfrac{v_k\cdot w_{k-1}}{w_{k-1}\cdot w_{k-1}} w_{k-1} + w_k$.

$$\Rightarrow [v_1 \cdots v_n] = [u_1 \cdots u_n]\begin{bmatrix} \|w_1\| & & \\ & \ddots & \\ & & \|w_n\| \end{bmatrix}\begin{bmatrix} 1 & & \frac{v_i\cdot w_j}{w_j w_j} \\ & \ddots & \\ & & 1 \end{bmatrix}$$

and as we proved above:

$$\left|\frac{v_i\cdot w_j}{w_j\cdot w_j}\right| \le \frac{1}{2} \quad \text{and} \quad \|w_i\| / \|w_{i+1}\| \le 2/\sqrt{3}.$$

$$\Rightarrow [v_1 \cdots v_n] \in \Sigma_{2/\sqrt{3},\,1/2}.$$

② $\forall g \in SL_n(\mathbb{R})$, $g\mathbb{Z}^n = [v_1 \cdots v_n]\mathbb{Z}^n$ where $\{v_1, \ldots, v_n\}$ satisfies the above properties. So $\{-v_1, v_2, \ldots, v_n\}$ also satisfies those properties. Either $[v_1\, v_2 \cdots v_n] \in SL_n(\mathbb{R})$ or $[-v_1\, v_2 \cdots v_n] \in SL_n(\mathbb{R})$.

So w.l.o.g. we can and will assume $[v_1 \cdots v_n] \in SL_n(\mathbb{R})$.

As above $[v_1 \cdots v_n] = [u_1 \cdots u_n]\begin{bmatrix} \|w_1\| & & \\ & \ddots & \\ & & \|w_n\| \end{bmatrix}\begin{bmatrix} 1 & & \frac{v_i\cdot w_j}{w_j w_j} \\ & \ddots & \\ & & 1 \end{bmatrix}$

Comparing the sign of determinants, we get $[u_1 \cdots u_n] \in SO(n)$. ■

<u>Mahler's compactness criterion</u>

$$X \subseteq \Omega^{(1)}(\mathbb{R}^n) \text{ is precompact} \iff \exists\, \delta_0 > 0,\ \delta(X) \subseteq [\delta_0, \infty)$$

($\Leftarrow$) Since $\Theta : SL_n(\mathbb{R})/SL_n(\mathbb{Z}) \to \Omega^{(1)}(\mathbb{R}^n)$ is a homeomorphism, it is enough to show

$$\{ g\,SL_n(\mathbb{Z}) \in SL_n(\mathbb{R})/SL_n(\mathbb{Z}) \mid \delta(g\mathbb{Z}^n) \geq \delta_0 \}$$

is precompact.

By the proof of the above theorem,

$$g = k \begin{bmatrix} \|\omega_1\| & & \\ & \ddots & \\ & & \|\omega_n\| \end{bmatrix} n\, \gamma \quad \text{s.t.}$$

- $\|n - I\|_\infty \leq 1/2$
- $k \in SO(n)$
- $\gamma \in SL_n(\mathbb{Z})$

- $\|\omega_1\| = \delta(g\mathbb{Z}^n) \geq \delta_0$
- $\|\omega_i\| / \|\omega_{i+1}\| \leq \alpha := 2/\sqrt{3}$
- $\|\omega_1\| \cdot \|\omega_2\| \cdot \cdots \cdot \|\omega_n\| = 1$.

$$\overset{0}{\Longrightarrow} \|\omega_1\| \leq \alpha^{i-1} \|\omega_i\| \quad \text{and} \quad \|\omega_i\| \ll_\alpha \|\omega_n\|^{1/n-i+1}$$

$$\Longrightarrow \|\omega_1\|^n \ll_\alpha \|\omega_1\| \cdot \|\omega_2\| \cdot \cdots \cdot \|\omega_n\| = 1 \ll_\alpha \|\omega_n\|^{O(1)}$$

$$\overset{\mathrm{I}}{\Longrightarrow} 1 \ll_{\delta_0, n} \|\omega_1\| \ll_{\alpha, n} 1 \quad \text{and} \quad 1 \ll_{\alpha, n} \|\omega_n\|.$$

And $\|\omega_1\|^{n-1} \|\omega_n\| \ll_\alpha 1$

$$\overset{\mathrm{II}}{\Longrightarrow} \|\omega_n\| \ll_{\alpha, \delta_0, n} 1$$

$$\overset{0,\mathrm{I},\mathrm{II}}{\Longrightarrow} 1 \ll_{\alpha, \delta_0, n} \|\omega_i\| \ll_{\alpha, \delta_0, n} 1$$

$$\Longrightarrow k \begin{bmatrix} \|\omega_1\| & & \\ & \ddots & \\ & & \|\omega_n\| \end{bmatrix} n \quad \text{is in a compact set.} \quad \blacksquare$$

Lemma $\rho(a)^2 \, dk \, da \, dn$ is a Haar measure of $SL_n(\mathbb{R})$

where dk is a Haar measure of $SO(n)$,

da is a Haar measure of $A = \left\{ \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} \,\middle|\, \overset{a_i \in \mathbb{R}^+}{a_1 \cdots a_n = 1} \right\}$,

dn is a Haar measure of $N = \left\{ \begin{bmatrix} 1 & & n_{ij} \\ & \ddots & \\ & & 1 \end{bmatrix} \,\middle|\, n_{ij} \in \mathbb{R} \right\}$,

and $\rho(a)^2 = \prod_{1 \le i < j \le n} (a_i / a_j)$.

__Pf__. We know that $G = KB$ where $B = AN$.

$$\Rightarrow \quad \mu(f) := \int_K \int_B f(kb) \frac{\Delta_B(b)}{\Delta_G(b)} \, db \, dk$$

$$= \int_K \int_B f(kb) \, \Delta_B(b) \, db \, dk \qquad \text{is a Haar measure.}$$

- $B = AN \Rightarrow \quad \ell(h) = \int_A \int_N h(an) \frac{\Delta_N(n)}{\Delta_B(n)} \, dn \, da$

  is a left Haar measure

- $\mu(f) = \int_K \int_A \int_N f(kan) \frac{\Delta_B(an)}{\Delta_B(n)} \, dn \, da \, dk$

$$= \int_K \int_A \int_N f(kan) \, \Delta_B(a) \, dn \, da \, dk.$$

__Claim__ • $dn \, da$ is a left Haar measure.

• $\rho(a)^2 \, dn \, da$ is a right Haar measure.

Pf. $(a'n')(an) = a'a\, a^{-1}n'a\, n$

$$\Rightarrow d(a^{-1}n'a\ n)\, d(a'a) = dn\ da,$$

$$(an)(a'n') = (aa')\,(a'^{-1}na')(n')$$

$$\Rightarrow d(aa')\, d(a'^{-1}na'.n') = da\ d(a'^{-1}na')$$

$$= \rho(a')^{-1}\ da\ dn$$

$$\Rightarrow\ \rho(aa')^{2} d(aa')\, d(a'^{-1}na'n') = \rho(aa')^{2}\rho(a')^{-2}\ da\, dn$$

$$= \rho(a)^{2}\ da\, dn. \quad \blacksquare$$

So $\Delta_B(a) = \rho(a)^2 \Rightarrow \mu(f) = \int_K \int_A \int_N f(kan)\, \rho(a)^2 dn\ da\ dk$

is a Haar measure of $SL_n(\mathbb{R})$. $\blacksquare$

<u>Proof of the above theorem</u> Since $SL_n(\mathbb{R}) = \Sigma^{(1)}_{2/\sqrt{3},1/2} SL_n(\mathbb{Z})$, it is enough to show $\mu(\Sigma^{(1)}_{\alpha,\beta}) < \infty$. (why?).

$$\mu(\Sigma^{(1)}_{\alpha,\beta}) = \int_K \int_{A^{(1)}_\alpha} \int_{N_\beta} \rho(a)^2\ dn\ da\ dk$$

$$= \mathrm{vol}(K)\,.\,\mathrm{vol}(N_\beta)\,.\int_{A^{(1)}_\alpha} \rho(a)^2\ da.$$

- $A^{(1)} \xrightarrow{\varphi} \underbrace{\mathbb{R}^+ \times \cdots \times \mathbb{R}^+}_{n-1}$ is a group homomorphism.

$$\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \longmapsto (d_1/d_2, \ldots, d_{n-1}/d_n)$$

- $\ker(\varphi) = \{I\}$
- φ is a group homeomorphism.

$\int \rho(\ )^2\ d \qquad \int \quad x^{m_1} \cdots x^{m_{n-1}}\ dx_1 \cdots dx_{n-1} \qquad \geq 1$

$$\int_{A_\alpha^{(1)}} \rho(a)^2 \, da = \int_{[0,\alpha]^{n-1}} x_1^{m_1} \cdots x_{n-1}^{m_{n-1}} \cdot \frac{dx_1}{x_1} \cdots \frac{dx_{n-1}}{x_{n-1}} \qquad , \ m_i \in \mathbb{Z}^{\geq 1} ,$$

$$= \prod_{i=1}^{n-1} \int_0^\alpha x_i^{m_i - 1} \, dx_i$$

$$= \prod_{i=1}^{n-1} \left(\frac{x_i^{m_i}}{m_i} \Big|_0^\alpha \right) < \infty . \qquad \blacksquare$$