Hlawka's theorem

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Siegel Transform

Let $f:\mathbb{R}^n\to\mathbb{R}$ be a Riemann integrable function. Suppose

① $|f(\vec{x})|\le M_0$ ② $f(\vec{x})=0$ if $\|\vec{x}\|\ge r_0$.

For such a function, let

$$\hat{f}: SL_n(\mathbb{R})/SL_n(\mathbb{Z})\to\mathbb{R},$$

$$\hat{f}(g\Gamma):=\sum_{v\in\mathbb{Z}^n\setminus\{\vec{0}\}} f(gv).$$

Since $g\mathbb{Z}^n$ is discrete and f is compactly supported, $\hat{f}(g\Gamma)$ is finite.

For $\lambda\in(0,1]$, let $f_\lambda:\mathbb{R}^n\to\mathbb{R}$, $f_\lambda(\vec{x})=\lambda^n f(\lambda\vec{x})$.

Proposition $\exists\, F: G/\Gamma\to\mathbb{R}^{\ge 0}$, ① $\int_{G/\Gamma} F(g\Gamma)\,d\mu(g)<\infty$

② $\forall\lambda\in(0,1]$, $\widehat{|f_\lambda|}(g\Gamma)\le F(g\Gamma)$,

where μ is a probability G-invariant regular measure on G/Γ.

Proof. $\widehat{|f_\lambda|}(g\Gamma)=\lambda^n\sum_{v\in\mathbb{Z}^n\setminus 0}|f(\lambda g v)|$

$$\le M_0\cdot\lambda^n\cdot|\{v\in\mathbb{Z}^n \mid \|gv\|\le r_0^{-1}\lambda\}|.$$

W.L.O.G. we can and will assume $g \in \Sigma_{\alpha,\beta}$. So $g = kau$ where $k \in SO_n(\mathbb{R})$, $a = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix}$, $u = \begin{bmatrix} 1 & & u_{ij} \\ & \ddots & \\ & & 1 \end{bmatrix}$ s.t.

$$a_i / a_{i+1} \leq \alpha, \quad |u_{ij}| \leq \beta.$$

$$\Rightarrow \|gv\|^2 = \sum_{i=1}^{n} a_i^2 (v_i + u_{i,i+1} v_{i+1} + \cdots + u_{in} v_n)^2 \leq r_0^2 \lambda^{-2}$$

$$\Rightarrow \begin{cases} a_n |v_n| \leq r_0 \lambda^{-1} \\ a_{n-1} |v_{n-1} + u_{n-1,n} v_n| \leq r_0 \lambda^{-1} \\ \vdots \\ a_1 |v_1 + u_{12} v_2 + \cdots + u_{1n} v_n| \leq r_0 \lambda^{-1} \end{cases}$$

$\Rightarrow$ • there are at most $\left(\frac{1}{a_n} 2 r_0 \lambda^{-1} + 1\right)$-many possibilities for v_n

• Fixing $v_n, \ldots, v_{k+1}$, there are at most $\left(\frac{1}{a_k} 2 r_0 \lambda^{-1} + 1\right)$-many possibilities for v_k.

$$\Rightarrow \widehat{\mathbb{1}_{\lambda}}(g\Gamma) \leq M_0 \lambda^n \prod_{i=1}^{n} \left(\frac{1}{a_i} 2 r_0 \lambda^{-1} + 1\right)$$

$$= M_0 \prod_{i=1}^{n} \left(\frac{2r_0}{a_i} + \lambda\right)$$

$$\leq M_0 \prod_{i=1}^{n} \left(\frac{2r_0}{a_i} + 1\right) =: F(g\Gamma).$$

• It is enough to show the integrability property of F.

$$\int_{\Sigma^{(1)}} F(g)\, dg = \int_K dk \cdot \int_N dn \cdot \int_{A_\alpha^{(1)}} M_0 \cdot \prod_{i=1}^{n} \left(\frac{2r_0}{a_i} + 1\right) \cdot \rho(a)^2\, da.$$

$\Sigma^{(1)}_{\alpha,\beta}$ $\quad K \quad N_\beta \quad A^{(1)}_\alpha \quad \sum_{i=1} (a_i \ldots)$

Let $x_1 := a_1/a_2, \ldots, x_{n-1} := a_{n-1}/a_n$. So $a_k/a_n = x_k \cdot\ldots\cdot x_{n-1}$.

$$\Rightarrow 1 = a_1 \cdot a_2 \cdot \ldots \cdot a_n = (x_1 \cdot\ldots\cdot x_{n-1})\cdot(x_2 \cdot\ldots\cdot x_{n-1})\cdot\ldots\cdot(x_{n-1})\, a_n^n$$

$$\Rightarrow a_n = \prod_{i=1}^{n-1} x_i^{-i/n}.$$

And
$$\rho(a)^2 = \prod_{i<j} a_i/a_j = \prod_{i<j} (x_i \cdot x_{i+1} \cdot\ldots\cdot x_{j-1}) = \prod_{i=1}^{n-1} x_i^{i(n-i)}.$$

Hence
$$\prod_{i=1}^{n}\left(\frac{2r_0}{a_i}+1\right)\cdot\rho(a)^2\,da = \left[\prod_{i=1}^{n} a_n\cdot\left(\frac{2r_0}{a_n}+\frac{a_i}{a_n}\right)\right]\cdot\rho(a)^2\,da$$

$$= \prod_{i=1}^{n-1} x_i^{-i}\cdot\left(2r_0\prod_{1}^{n-1} x_j^{j/n}+1\right)\cdot\prod_{i=1}^{n-1}\left(2r_0\prod_{j=1}^{n-1} x_j^{j/n}+\prod_{j=i}^{n-1} x_j\right)\cdot\prod x_i^{i(n-i)}\,\frac{dx_1}{x_1}\cdots\frac{dx_{n-1}}{x_{n-1}}$$

So we should focus on the following integral:

$$\int_0^\alpha\cdots\int_0^\alpha\left(\prod_{i=1}^{n-1} x_i^{i(n-i)-i-1}\right)\cdot x_{n-1}^{(n-1)^2/n}\cdot\prod_{i=1}^{n-2}\left(2r_0\prod_{1}^{n-2} x_j^{j/n}+\prod_{i}^{n-2} x_j\right)dx_1\cdots dx_{n-1}$$

$$\ll_{n,\alpha}\left(\prod_{i=1}^{n-2}\int_0^\alpha x_i^{i(n-i-1)-1}\,dx_i\right)\cdot\int_0^\alpha x_{n-1}^{\frac{(n-1)^2}{n}-1}\,dx_{n-1}.$$

$$\ll_{\alpha,n}\int_0^\alpha x^{\frac{(n-1)^2}{n}-1}\,dx.$$

- If $n\geq 3$, then $\frac{(n-1)^2}{n}-1>0$.
- If $n=2$, then $\int_0^\alpha x^{-1/2}\,dx = 2\sqrt{x}\,\Big|_0^\alpha = 2\sqrt{\alpha}<\infty$. ∎

Corollary. In the above setting, $f \in L^1(G/\Gamma)$.

Corollary. In the above setting,

$$\lim_{\lambda\to 0} \int_{G/\Gamma} \hat{f}_\lambda \, d\mu(g) = \int_{\mathbb{R}^n} f(x)\, dx,$$

where μ is the probability "Haar" measure on G/Γ.

Proof. By dominant theorem and the above proposition,

$$\lim_{\lambda\to 0} \int_{G/\Gamma} \hat{f}_\lambda(g\Gamma)\, d\mu = \int_{G/\Gamma} \lim_{\lambda\to 0} \hat{f}_\lambda(g\Gamma)\, d\mu(g).$$

And since f is Riemann integrable

$$\lim_{\lambda\to 0} \hat{f}_\lambda(g\Gamma) = \lim_{\lambda\to 0} \lambda^n \sum_{v\in\mathbb{Z}^n\setminus\vec{0}} f(\lambda g v) = \int_{\mathbb{R}^n} f(x)\, dx.$$

One gets the claim using $\int_{G/\Gamma} d\mu(g) = 1$. ∎

$$\int_{G/\Gamma} \hat{f}_\lambda(g\Gamma)\, d\mu(g) = \int_{G/\Gamma} \lambda^n \sum_{v\in\mathbb{Z}^n\setminus 0} f(\lambda g v)\, d\mu(g)$$

$$= \sum_{m=1}^{\infty} \frac{1}{m^n} \int_{G/\Gamma} \sum_{\substack{v\in\mathbb{Z}^n \\ \text{primitive}}} (m\lambda)^n f(m\lambda g v)\, d\mu(g)$$

$$= \sum_{m=1}^{\infty} \frac{1}{m^n} \int_{G/\Gamma} \sum_{\Gamma/\Gamma_{e_1}} f_{m\lambda}(g\gamma e_1)\, d\mu(g)$$

$$\sum^{\infty} \; 1 \int f$$

$$= \sum_{m=1}^{\infty} \frac{1}{m^n} \int_{G/\Gamma_{e_1}} f_{m\lambda}(g e_1)\, d\mu'(g) \qquad \circledast$$

where $\Gamma_{e_1} = \left\{ \left[\begin{array}{c|c} 1 & v \\ \hline 0 & \gamma_{n-1} \end{array} \right] \,\middle|\, v \in \mathbb{Z}^{n-1},\ \gamma_{n-1} \in SL_{n-1}(\mathbb{Z}) \right\}$.

In order to compute the above integral inductive, we introduce the following Haar measure on $SL_n(\mathbb{R})$:

For a measurable subset $A \subseteq SL_n(\mathbb{R})$, let

$$\mu(A) := \ell(C_A), \text{ where } C_A := \{ \lambda g \mid \lambda \in (0,1],\ g \in A \} \text{ and}$$

ℓ is the Lebesgue measure on $M_n(\mathbb{R})$.

<u>Lemma</u>. μ is a Haar measure.

<u>Proof</u>. It is clear that μ is regular. So it is enough to check that μ is $SL_n(\mathbb{R})$-invariant.

$$\mu(gA) = \ell(C_{gA}) = \ell(g C_A) = \det(g)^n \, \ell(C_A)$$
$$= \ell(C_A) = \mu(A). \quad \blacksquare$$

<u>Remark</u>. $C_{SL_n(\mathbb{R})} = \{ g \in M_n(\mathbb{R}) \mid 0 < \det(g) \leq 1 \}$, and

- $SL_n(\mathbb{R}) \curvearrowright C_{SL_n(\mathbb{R})}$ by left multiplication.
- $C_{SL_n(\mathbb{R})} \curvearrowleft SL_n(\mathbb{R})$ by right multiplication.

· $C_{SL_n(\mathbb{R})} \curvearrowleft GL_n(\mathbb{R})$ by right multiplication.

<u>Lemma</u>. $\forall f: SL_n(\mathbb{R}) \to \mathbb{R}$ integrable

$$\int_{SL_n(\mathbb{R})} f(g)\, d\mu(g) = \int_{C_{SL_n(\mathbb{R})}} f(\det(y)^{-1/n} y)\, dy.$$

<u>Pf</u>. It is enough to show this for characteristic function of finite measure sets.

$$\int_{SL_n(\mathbb{R})} 1_A(g)\, d\mu(g) = \mu(A) = \ell(C_A)$$

$$= \int_{C_{SL_n(\mathbb{R})}} 1_{C_A}(y)\, dy.$$

$$\boxed{\begin{aligned} [y \in C_A] &= [\exists \lambda \in (0,1],\ \exists y' \in A,\ \lambda y' = y] \\ &= [y \in C_{SL_n(\mathbb{R})},\ \det(y)^{-1/n} y \in A] \end{aligned}}$$

$$= \int_{C_{SL_n(\mathbb{R})}} 1_A(\det(y)^{-1/n} y)\, dy. \qquad \blacksquare$$

<u>Lemma</u>. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two fundamental regions of Γ_{e_1}, then for any function $\xi : G \to \mathbb{R}$ which factors through Γ_{e_1} we have

$$\int_{\mathcal{F}_1} \xi(\det(Y)^{-1/n} Y)\, dY = \int_{\mathcal{F}_2} \xi(\det(Y)^{-1/n} Y)\, dY.$$

<u>Pf</u>: $$\int_{\mathcal{F}_1} \xi(\det(Y)^{-1/n} Y)\, dY = \sum_{\gamma \in \Gamma_{e_1}} \int_{\mathcal{F}_1 \cap \mathcal{F}_2 \gamma} \xi(\det(Y)^{-1/n} Y)\, dY$$

$$= \sum_{\gamma \in \Gamma'_{e_1}} \int_{\mathcal{F}_1\gamma^{-1}\cap \mathcal{F}_2} \xi\left(\det(Y)^{-1/n} \det(\gamma)^{-1/n} Y\gamma\right) (\det \gamma)^n \, dY$$

$\boxed{\begin{array}{l} \det \gamma = 1 \\ \xi \text{ is } \Gamma'_{e_1}\text{-inv.} \end{array}}$

$$= \sum_{\gamma \in \Gamma'_{e_1}} \int_{\mathcal{F}_1\gamma^{-1}\cap \mathcal{F}_2} \xi\left(\det(Y)^{-1/n} Y\right) dY$$

$$= \int_{\cup(\mathcal{F}_1\gamma^{-1}\cap \mathcal{F}_2)} \xi\left(\det(Y)^{-1/n} Y\right) dY$$

$$= \int_{\mathcal{F}_2} \xi\left(\det(Y)^{-1/n} Y\right) dY . \qquad \blacksquare$$

So to further simplify ⊛ we need to find a "good" fundamental region of Γ_{e_1} in $C_{SL_n(\mathbb{R})}$.

- Let $\mathbb{R}^n \setminus \{0\} \longrightarrow SL_n(\mathbb{R})$, $x \mapsto g_x$ be a (measurable) section of $g \mapsto g\vec{e}_1$.
- So any $Y \in C_{SL_n(\mathbb{R})}$ can be written as $g_x \begin{bmatrix} 1 & v \\ 0 & Y' \end{bmatrix}$.
- $\begin{bmatrix} 1 & v \\ & Y' \end{bmatrix}\begin{bmatrix} 1 & \omega \\ & \gamma' \end{bmatrix} = \begin{bmatrix} 1 & \omega + v\gamma' \\ & Y'\gamma' \end{bmatrix} = \begin{bmatrix} 1 & (\omega\gamma'^{-1} + v)\gamma' \\ & Y'\gamma' \end{bmatrix}$

So if $\mathcal{F}_{n-1}$ is a fundamental region of $SL_{n-1}(\mathbb{Z})$ in $SL_{n-1}(\mathbb{R})$, then $Y' \in C_{\mathcal{F}_{n-1}}$ and $v \in [0,1)^{n-1}$ gives us a fundamental region in the above action. And, since $\det(g_x) = 1$, the volume form in the above coordinates is $d^n x\, dv\, dY'$. So

$$\int_{G/\Gamma_{e_1}} f_{m\lambda}(g e_1)\,dg = \int_{\mathbb{R}^n}\int_{[0,1)^{n-1}}\int_{C_{\mathcal{F}_{n-1}}} f_{m\lambda}\left(\det\left(g_x\begin{bmatrix}1 & v\\ 0 & Y'\end{bmatrix}\right)^{-1/n} x\right) dY'\,dv\,dx$$

$$= \int_{C_{\mathcal{F}_{n-1}}}\int_{\mathbb{R}^n}\int_{[0,1)^{n-1}} f_{m\lambda}\left(\det(Y')^{-1/n} x\right) dv\,dx\,dY'$$

$$= \int_{C_{\mathcal{F}_{n-1}}}\int_{\mathbb{R}^n} (m\lambda)^n f\big(\underbrace{(m\lambda)\det(Y')^{-1/n} x}_{x'}\big)\,dx\,dY'$$

$$\rightsquigarrow dx' = (m\lambda)^n \det(Y')^{-1}\,dx$$

$$= \int_{C_{\mathcal{F}_{n-1}}}\int_{\mathbb{R}^n} \det(Y')\, f(x')\,dx'\,dY'$$

$$= \int_{\mathbb{R}^n} f(x)\,dx \cdot \int_{C_{\mathcal{F}_{n-1}}}\int_0^{\det(Y')} dr\,dY'$$

$$= \int_{\mathbb{R}^n} f(x)\,dx \cdot \int_{\substack{\{(Y',r)\,\mid\, 0\le r\le \det(Y')\le 1\} \\ Y'\in C_{\mathcal{F}_{n-1}}}} dY'\,dr$$

$$= \int_{\mathbb{R}^n} f(x)\,dx \cdot \int_0^1 \Big(\underbrace{v_{n-1}}_{\mathrm{vol}(SL_{n-1}(\mathbb{R})/SL_{n-1}(\mathbb{Z}))} - \int_{r^{1/n-1}C_{\mathcal{F}_{n-1}}} dY'\Big)\,dr$$

$$= \int_{\mathbb{R}^n} f(x)\,dx \cdot \int_0^1 \left(v_{n-1} - r^{n-1} v_{n-1}\right) dr$$

$$= \int_{\mathbb{R}^n} f(x)\,dx \cdot \left(r - \frac{r^n}{n}\right)\Big|_0^1 \cdot v_{n-1}$$

$$\mathbb{R}^n$$

$$= \int_{\mathbb{R}^n} f(x)\,dx \cdot \frac{n-1}{n} \cdot v_{n-1} \cdot \qquad (**)$$

$$(*),(**) \Rightarrow \int_{G/\Gamma} \hat{f}_\lambda(g\Gamma)\,dg = \zeta(n) \cdot \frac{n-1}{n} \cdot v_{n-1} \cdot \int_{\mathbb{R}^n} f(x)\,dx \qquad (\dagger)$$

Now, letting $\lambda \to 0^+$, we get

$$v_n \cdot \int_{\mathbb{R}^n} f(x)\,dx = \zeta(n) \cdot \frac{n-1}{n} \cdot v_{n-1} \cdot \int_{\mathbb{R}^n} f(x)\,dx$$

$$\Rightarrow \quad n\, v_n = (n-1)\, v_{n-1} \cdot \zeta(n)$$

$$\Rightarrow \quad v_n = \frac{1}{n} \zeta(2) \cdot \zeta(3) \cdot \cdots \cdot \zeta(n) \cdot$$

$(\dagger)$ implies $\int_{G/\Gamma} \hat{f}_\lambda(g\Gamma)\,dg$ is independent of $\underline{\lambda}$. So

if μ is the prob. Haar measure on G/Γ, then

$$\int_{G/\Gamma} \hat{f}(g\Gamma)\,d\mu = \int_{\mathbb{R}^n} f(x)\,dx .$$

Moreover $$\int_{G/\Gamma} \hat{f}(g\Gamma)\,d\mu = \sum_{m=1}^{\infty} \frac{1}{m^n} \underbrace{\int_{G/\Gamma} \sum_{\substack{v\in\mathbb{Z}^n \\ \text{primitive}}} m^n f(mgv)\,dg}_{\int \sum_{\substack{v\in\mathbb{Z}^n \\ \text{primitive}}} f(gv)\,dg}$$

$$\int_{G/\Gamma} \sum_{\substack{v\in\mathbb{Z}^n \\ \text{primitive}}} f(gv)\, dg$$

$$\Rightarrow \int_{\mathbb{R}^n} f(x)\,dx = \zeta(n) \int_{G/\Gamma} \sum_{\substack{v\in\mathbb{Z}^n \\ \text{primi.}}} f(gv)\, dg. \qquad (\ast\ast\ast)$$

<u>Corollary</u>. $A \subseteq \mathbb{R}^n$ bounded, $\ell(A) < \zeta(n)$

$$\Rightarrow \exists \Delta \in \Omega^{(1)}(\mathbb{R}^n), \quad \Delta \cap A \subseteq \{0\}.$$

<u>Proof</u>. Let $f = 1_A$. So by $(\ast\ast\ast)$

$$\ell(A) = \zeta(n) \int_{G/\Gamma} \sum_{\substack{v\in\mathbb{Z}^n \\ \text{primi.}}} 1_A(gv)\, dg.$$

$$\Rightarrow 1 > \frac{\ell(A)}{\zeta(n)} = \int_{G/\Gamma} \left|\begin{matrix}\text{primitive}\\ \text{vectors}\end{matrix} \text{ of } g\mathbb{Z}^n \cap A\right| dg$$

$$\Rightarrow \exists g \in G, \quad 1 > \left|\begin{matrix}\text{prim.}\\ \text{vect.}\end{matrix} \text{ of } g\mathbb{Z}^n \cap A\right|$$

$$\Rightarrow \exists g \in G, \; g\mathbb{Z}^n \cap A \subseteq \{0\}. \quad \blacksquare$$