Lecture 4: Inner product

Wednesday, September 28, 2016 8:50 AM

Ex. Let $\vec{v} = (t, 2t, 1)$. Then

(a) $\vec{v} \cdot (-2\vec{i} + \vec{j}) = 0$ for any t

(b) Find all t such that $\vec{v} \cdot (\vec{i} + \vec{j} + \vec{k}) = 0$.

Solution. (a) $\vec{v} \cdot (-2\vec{i} + \vec{j}) = -2\, \underbrace{\vec{v} \cdot \vec{i}}_{1^{st} \text{ component of } \vec{v}} + \underbrace{\vec{v} \cdot \vec{j}}_{2^{nd} \text{ comp. of } \vec{v}}$

distribution rule

$$= -2t + 2t = 0.$$

(b) $\vec{v} \cdot (\vec{i} + \vec{j} + \vec{k}) = \vec{v} \cdot \vec{i} + \vec{v} \cdot \vec{j} + \vec{v} \cdot \vec{k}$

$$= t + 2t + 1 = 3t + 1 = 0$$

$$\Rightarrow t = -1/3.$$

Basic algebraic properties:

ⓐ $(c\vec{v}) \cdot \vec{w} = c\, \vec{v} \cdot \vec{w} = \vec{v} \cdot (c\vec{w})$

ⓑ $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$

ⓒ $\vec{v} \cdot (\vec{w}_1 + \vec{w}_2) = \vec{v} \cdot \vec{w}_1 + \vec{v} \cdot \vec{w}_2$.

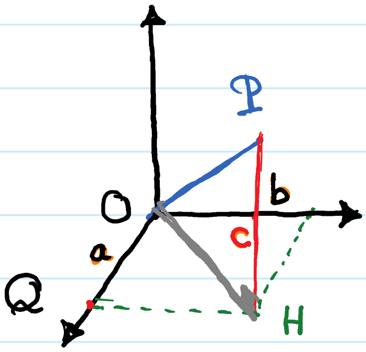
Next we will see Geometric Properties of inner product.

Lecture 4: Length and direction of a 3D vector

Friday, September 30, 2016 8:10 AM

Before we talk about geometric properties of inner product, let's see how we can compute length of a 3D vector, and how we can encode "direction" of a 3D vector.

Length of a 3D vector.



$$OP^2 = PH^2 + OH^2 \quad (\text{since } \hat{PHO} \text{ is a right angle triangle.})$$

$$= c^2 + OQ^2 + QH^2 \quad (\text{since } \hat{HQO} \text{ is a right angle triangle})$$

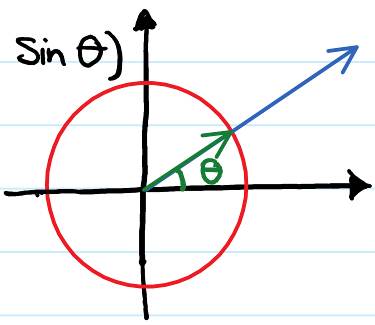
$$= c^2 + a^2 + b^2$$

So, if $\vec{v} = (a,b,c)$, then

$$\|\vec{v}\| = \sqrt{a^2+b^2+c^2}.$$

How about direction? In 2D we used an angle to encode direction, but this does NOT work in 3D. Let's go back to 2D: we found out

$$\vec{v} = (r\cos\theta, r\sin\theta) = r(\cos\theta, \sin\theta)$$



So $\vec{v} = \|\vec{v}\| \underbrace{(\cos\theta, \sin\theta)}_{\text{The vector of length 1 in the direction of } \vec{v}}$.

Lecture 4: The unit vector in the direction of a vector

Friday, September 30, 2016 8:30 AM

So the best way of encoding "the direction" of a given vector $\vec{v}$ is using the unit vector $\vec{e}_{\vec{v}}$ in the same direction as $\vec{v}$, in which case we have

$$\vec{v} = \|\vec{v}\| \, \vec{e}_{\vec{v}} \, .$$

Therefore we get

> The unit vector $\vec{e}_{\vec{v}}$ in the direction of $\vec{v}$ is given by $\dfrac{\vec{v}}{\|\vec{v}\|}$. And we say that we have normalized $\vec{v}$.

Ex. Find the unit vector in the direction of $\vec{v} = (1,2,-3)$.

Solution. $\|(1,2,-3)\| = \sqrt{(1)^2+(2)^2+(-3)^2} = \sqrt{1+4+9}$

$= \sqrt{14}$.

So $\vec{e}_{\vec{v}} = \dfrac{1}{\sqrt{14}}(1,2,-3) = \left(\dfrac{1}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}, \dfrac{-3}{\sqrt{14}}\right)$.

Ex: Normalize $\vec{v} = (3,0,-4)$.

Solution. We have to find $\dfrac{\vec{v}}{\|\vec{v}\|}$. Let's start with

$\|\vec{v}\| = \sqrt{(3)^2+(0)^2+(-4)^2} = \sqrt{9+0+16} = 5$. So $\dfrac{\vec{v}}{\|\vec{v}\|} = \left(\frac{3}{5}, 0, \frac{4}{5}\right)$.

Lecture 4: Geometric properties of inner product

Friday, September 30, 2016 8:50 AM

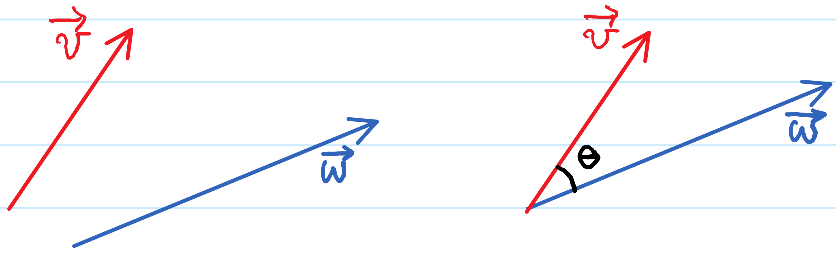
Let $\vec{v} = (a, b, c)$.

$$\vec{v}\cdot\vec{v} = (a)(a) + (b)(b) + (c)(c) = a^2 + b^2 + c^2 = \|\vec{v}\|^2$$

This gives us the first geometric property of inner product

$$\boxed{\vec{v}\cdot\vec{v} = \|\vec{v}\|^2}$$

Let $\vec{v}$ and $\vec{w}$ be two vectors. Draw them in a way that they have a <u>common initial point</u>.



The angle between these segments is called <u>the angle between $\vec{v}$ and $\vec{w}$</u>.

Here is the extremely important property of inner product:

$$\boxed{\vec{v}\cdot\vec{w} = \|\vec{v}\|\,\|\vec{w}\|\cos\theta}$$

<u>why?</u>