Lecture 6: Determinant

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In the previous lecture we defined the determinant of a 2×2 matrix $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

Determinant of a 3×3 matrix is defined as follows:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = +a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix} \qquad \begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix} \qquad \begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix}$$

<u>Ex.</u> $$\begin{vmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 6 & 5 & 3 \end{vmatrix} = 1 \begin{vmatrix} 2 & 0 \\ 5 & 3 \end{vmatrix} - 0 \begin{vmatrix} 4 & 0 \\ 6 & 3 \end{vmatrix} + 0 \begin{vmatrix} 4 & 2 \\ 6 & 5 \end{vmatrix}$$

$$= 1\big((2)(3) - (0)(5)\big) = 6.$$

(In fact $\begin{vmatrix} a_1 & 0 & 0 \\ b_1 & a_2 & 0 \\ c_1 & b_2 & a_3 \end{vmatrix} = a_1 a_2 a_3$.)

<u>Ex.</u> $$\begin{vmatrix} a & b & c \\ x & y & z \\ x & y & z \end{vmatrix} = a \begin{vmatrix} y & z \\ y & z \end{vmatrix} - b \begin{vmatrix} x & z \\ x & z \end{vmatrix} + c \begin{vmatrix} x & y \\ x & y \end{vmatrix}$$

$$= (a)(0) - (b)(0) + (c)(0)$$

$$= 0.$$

Lecture 6: Determinant 3x3, cross product

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Ex. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 1 \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} - 1 \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + 1 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$

$= (18-12) - (9-4) + (3-2)$

$= 2.$

Remark. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (c-a)(c-b)(b-a)$

It is a special case of Vandermonde's determinant.

Ex. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$

$= (45-48) - 2(36-42) + 3(32-35)$

$= -3 - 2 \times (-6) + 3 \times (-3)$

$= 0.$

You will see the importance of determinant in linear algebra. In this course we use it to understand cross product and its geometric properties.

Definition Suppose $\vec{v} = (x_1, y_1, z_1)$ and $\vec{w} = (x_2, y_2, z_2)$.

Then cross product of $\vec{v}$ and $\vec{w}$ is $\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$

Lecture 6: Cross product

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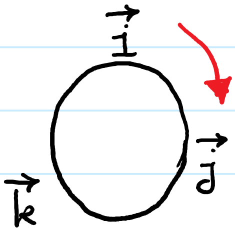
It is a symbolic determinant, which means

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \vec{i} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \vec{j} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \vec{k}.$$

Ex. $\vec{i} \times \vec{j} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \vec{k}$

$= \vec{k}$

Remark.



$\vec{i} \times \vec{j} = \vec{k}$

$\vec{j} \times \vec{k} = \vec{i}$

$\vec{k} \times \vec{i} = \vec{j}$

Ex. $\vec{v} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ x & y & z \end{vmatrix} = (0,0,0) = \vec{0}$.

Algebraic Properties of Cross Product.

1. $\vec{v} \times (\vec{\omega}_1 + \vec{\omega}_2) = \vec{v} \times \vec{\omega}_1 + \vec{v} \times \vec{\omega}_2$ } distribution
2. $(\vec{v}_1 + \vec{v}_2) \times \vec{\omega} = \vec{v}_1 \times \vec{\omega} + \vec{v}_2 \times \vec{\omega}$
3. $(c\,\vec{v}) \times \vec{\omega} = \vec{v} \times (c\,\vec{\omega}) = c(\vec{v} \times \vec{\omega})$.
4. $\vec{v} \times \vec{\omega} = -\vec{\omega} \times \vec{v}$.

Lecture 6: Algebraic properties of cross product

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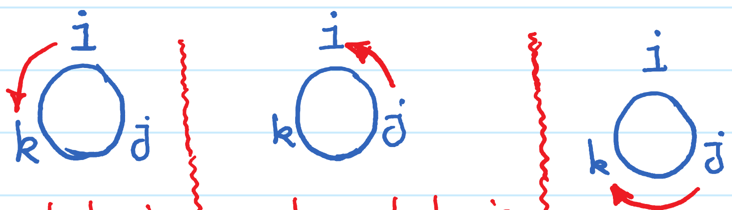
Using algebraic properties and "the $\vec{i}, \vec{j}, \vec{k}$ wheel" we can compute cross product without determinant.

Ex. Find $(2\vec{i}+\vec{j})\times(\vec{i}-3\vec{k})$.

Solution. $(2\vec{i}+\vec{j})\times(\vec{i}-3\vec{k})$

$$= (2\vec{i})\times\vec{i} + (2\vec{i})\times(-3\vec{k}) + \vec{j}\times\vec{i} + \vec{j}\times(-3\vec{k})$$

$$= 2\,\underbrace{\cancel{\vec{i}\times\vec{i}}}_{\vec{0}} - 6\,\vec{i}\times\vec{k} + \vec{j}\times\vec{i} - 3\,\vec{j}\times\vec{k}$$



Counter clockwise So $-\vec{j}$. | Counter clockwise So $-\vec{k}$ | clockwise so $\vec{i}$

$$= 6\vec{j} - \vec{k} - 3\vec{i}.$$

Ex. Suppose $\vec{v}\times\vec{w} = (1,2,3)$. Find $(2\vec{v}-\vec{w})\times(\vec{v}+3\vec{w})$.

Remark. In this example, you see that knowing $\vec{v}\times\vec{w}$ one can compute cross product of any two linear combinations of $\vec{v}$ and $\vec{w}$.

Solution. $(2\vec{v}-\vec{w})\times(\vec{v}+3\vec{w}) = 2\,\cancel{\vec{v}\times\vec{v}} + 6\,\vec{v}\times\vec{w} - \vec{w}\times\vec{v} - 3\,\cancel{\vec{w}\times\vec{w}}$

$$= 6\,\vec{v}\times\vec{w} + \vec{v}\times\vec{w} = 7\,\vec{v}\times\vec{w} = (7,14,21).$$

Lecture 6: Geometric properties of cross product

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$\vec{v}\times\vec{w}$ is a vector and any vector carries two information: direction and length.

To understand direction of $\vec{v}\times\vec{w}$ we start with a dot product computation:

Let $\vec{v}=(a_1,b_1,c_1)$, $\vec{w}=(a_2,b_2,c_2)$, and $\vec{u}=(a_3,b_3,c_3)$.

Then

$$(\vec{v}\times\vec{w})\cdot\vec{u}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}\cdot(a_3,b_3,c_3)$$

$$=\left(\begin{vmatrix}b_1 & c_1\\ b_2 & c_2\end{vmatrix}\vec{i}-\begin{vmatrix}a_1 & c_1\\ a_2 & c_2\end{vmatrix}\vec{j}+\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}\vec{k}\right)\cdot(a_3,b_3,c_3)$$

$$=\begin{vmatrix}b_1 & c_1\\ b_2 & c_2\end{vmatrix}a_3-\begin{vmatrix}a_1 & c_1\\ a_2 & c_2\end{vmatrix}b_3+\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}c_3$$

$$=\begin{vmatrix}a_3 & b_3 & c_3\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}=-\begin{vmatrix}a_1 & b_1 & c_1\\ a_3 & b_3 & c_3\\ a_2 & b_2 & c_2\end{vmatrix}$$

Hence

$$\boxed{(\vec{v}\times\vec{w})\cdot\vec{u}=\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}}$$