Lecture 11: Limit

Monday, October 17, 2016 8:50 AM

In the previous lecture we learned 3 rules for limits of multivariable functions:

Rule 1. Function is nice; You plug in and no problem arises; $\Rightarrow$ it is the answer.

Rule 2. $\left.\begin{array}{l}\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = t_0 \\ \displaystyle\lim_{t\to t_0} g(t) = L\end{array}\right\} \Rightarrow \displaystyle\lim_{(x,y)\to(a,b)} g(f(x,y)) = L.$

Rule 3. Approach (a,b) via various lines and check if

$$\lim_{x\to a} f(x, k(x-a)+b) \text{ depends on } k \text{ or NOT.}$$

$y-b=k(x-a)$

If it depends on k, then $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)$ does NOT exist.

Warning. If in Rule 3 the considered limit is independent of k, you cannot conclude anything.

Rule 4. Try other curves to approach (a,b). A good place to start for approaching $(0,0)$ is considering curves of the form $x=y^c$ or $y=x^c$.

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Ex. Determine if the following limit exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6+y^2}.$$

Solution. [Clearly we cannot use rule 1 and rule 2]

Approaching $(0,0)$ along the line $y=kx$:

$$\lim_{x\to 0} \frac{x^3(kx)}{x^6+k^2x^2} = \lim_{x\to 0} \frac{kx^2}{x^4+k^2} = 0,$$

Since it is independent of k, we cannot conclude anything.

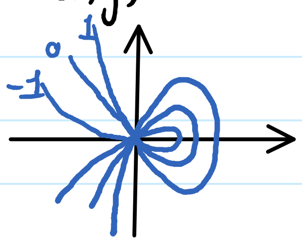
Let's approach $(0,0)$ along a curve of the form $y=x^c$.

We choose c such that numerator and denominator have the same degree: $c=3$.

$$\lim_{x\to 0} \frac{x^3\cdot x^3}{x^6+(x^3)^2} = \frac{1}{2}.$$

Since $\frac{1}{2}\neq 0$, $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3y}{x^6+y^2}$ does NOT exist.

Ex. Suppose the contour diagram of $f(x,y)$ looks like



Determine whether $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist.

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Solution Limit does NOT exist since we can approach $(0,0)$ along different level curves. Along the curve $f(x,y)=1$, the limit is 1, and along the curve $f(x,y)=-1$ the limit is -1. So $\lim_{(x,y)\to(0,0)} f(x,y)$ does NOT exist.

If two level curves $f(x,y)=c_1$ and $f(x,y)=c_2$ can approach to the point (a,b), then

$$\lim_{(x,y)\to(a,b)} f(x,y) \text{ does NOT exist.}$$

Rule 3 and 4 are useful to show a limit does NOT exist.

Rule 5. Use Squeeze Theorem:

If $g_1(x,y) \le f(x,y) \le g_2(x,y)$ and

$$\lim_{(x,y)\to(a,b)} g_1(x,y) = \lim_{(x,y)\to(a,b)} g_2(x,y) = L,$$

then

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

Ex. Determine if the following exists. If it does, find its value.

$$\lim_{(x,y)\to(0,0)} \sin(xy)\cos\left(\frac{1}{x^2+y^2}\right).$$

Solution. When we try to plug in, we realize that

$$\lim_{(x,y)\to(0,0)} \sin(xy) = 0 \quad , \quad \text{but} \quad \lim_{(x,y)\to(0,0)} \cos\left(\frac{1}{x^2+y^2}\right)$$

does NOT exist.

We have

$$\left|\sin(xy)\cos\left(\frac{1}{x^2+y^2}\right)\right| \le |\sin(xy)|$$

as $\left|\cos\left(\frac{1}{x^2+y^2}\right)\right| \le 1$. Hence

$$-|\sin(xy)| \le \sin(xy)\cos\left(\frac{1}{x^2+y^2}\right) \le |\sin(xy)|$$

and $\lim_{(x,y)\to(0,0)} \pm|\sin(xy)| = 0$. Therefore by the squeeze theorem $\lim_{(x,y)\to(0,0)} \sin(xy)\cos\left(\frac{1}{x^2+y^2}\right) = 0$.

By a similar argument we have:

If $\lim_{(x,y)\to(a,b)} f(x,y) = 0$ and $g(x,y)$ is bounded,

then $\lim_{(x,y)\to(a,b)} f(x,y)\,g(x,y) = 0$.

Important method : using polar coordinates.

First notice $\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(0,0)} f(x+a, y+b)$.

So one has to understand limits where (x,y) approaches $(0,0)$.

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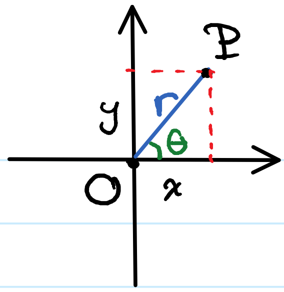

Writing x and y in polar coordinates means

$x = r\cos\theta$ and $y = r\sin\theta$ where

r is the distance of (x,y) from the origin, i.e. $r = \sqrt{x^2+y^2}$

and θ is the angle that the segment OP makes with the x-axis. So, as $(x,y) \longrightarrow (0,0)$, $r \longrightarrow 0$.

Hence
$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{\substack{r\to 0 \\ \text{uniform} \\ \text{on } \theta}} f(r\cos\theta, r\sin\theta)$$

The best way to understand the phrase "uniform on θ" is thinking about θ as <u>an unknown function</u> of r.

So we end up with reducing the two-variable limit to a single-variable limit with the caveat that we do not know what $\theta(r)$ is:

$$\boxed{\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} f(r\cos\theta(r), r\sin\theta(r)).}$$

We will see a general example of this type next time.