Lecture 13: Tangent plane

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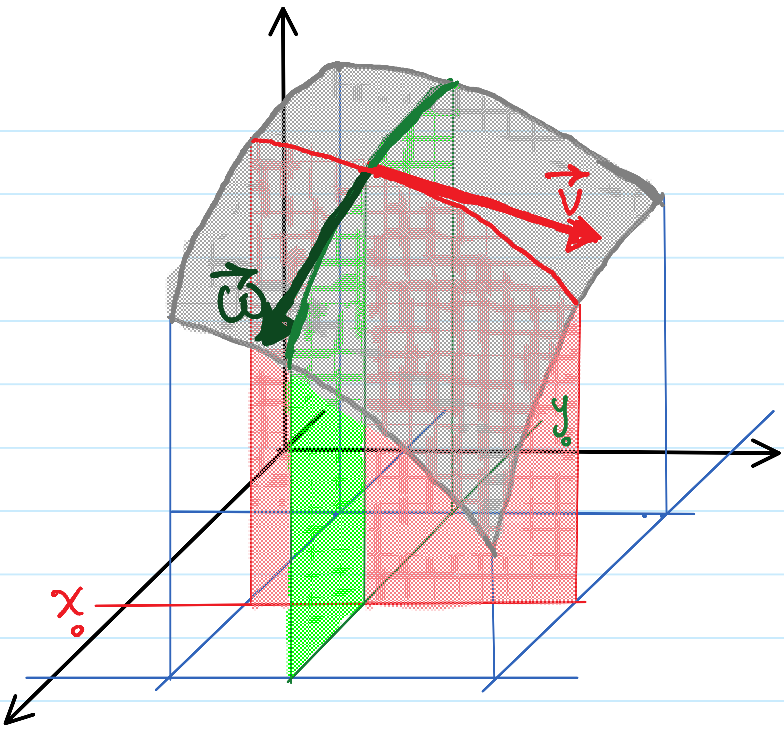
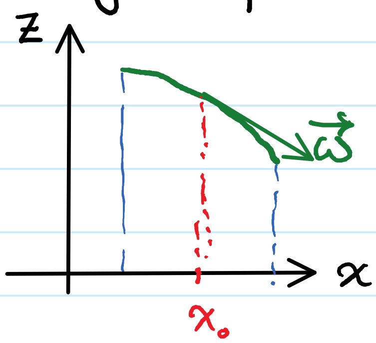


In the previous lecture we saw that $f_x(x_0,y_0)$ gives the "slope" of the tangent line of the green curve.

We also discussed that in order to find an equation of the tangent plane at $(x_0, y_0, f(x_0,y_0))$, it is enough to find $\vec{v}$ and $\vec{w}$, and compute their cross product.

Let's focus on the green plane, i.e. $\underline{y=y_0}$.

$\vec{\omega}$ is in the direction of the tangent line which is



$$z-z_0 = f_x(x_0,y_0)\underbrace{(x-x_0)}_{t}$$

and $y=y_0$.

So $(x,y,z) = (x_0,y_0,z_0) + (1,0,f_x(x_0,y_0))\,t$.

Hence we can have $\boxed{\vec{\omega} = (1,0,f_x(x_0,y_0))}$

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Similarly we can have $\boxed{\vec{v} = (0, 1, f_y(x_0, y_0))}$

Now to find a normal vector of the tangent plane (if it exists)

we can compute $\vec{n}_0 = \vec{v} \times \vec{w}$

$$= (1, 0, f_x(x_0, y_0)) \times (0, 1, f_y(x_0, y_0))$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x(x_0, y_0) \\ 0 & 1 & f_y(x_0, y_0) \end{vmatrix}$$

$$\boxed{\vec{n}_0 = (-f_x(x_0, y_0), -f_y(x_0, y_0), 1).}$$

Therefore equation of the tangent plane at the point (x_0, y_0, z_0) is

$$-f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) + (z - z_0) = 0$$

So

$$\boxed{z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$

Ex. Find equation of tangent plane of $z = x^2 + y^2$ at $(1, 1, 2)$.
(If it exists!)

Solution. $f_x(x, y) = 2x$, $f_y(x, y) = 2y$.

Equation of the tangent plane is

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$$-f_x(1,1)\ (x-1) - f_y(1,1)(y-1) + (z-2) = 0$$

So $-2(x-1)-2(y-1)+z-2=0$. Thus

$$-2x-2y+z = -2-2+2 = -2.$$

<u>Ex.</u> Suppose $z = x^2 - y^2$ has tangent plane at all of its points.

Find the points on this surface at which $\vec{n}_0 = (3,1,2)$ is a normal vector of the tangent plane.

<u>Solution</u>. We know that at the point (x_0, y_0, z_0) normal vector of the tangent plane is parallel to $(-f_x(x_0,y_0), -f_y(x_0,y_0), 1)$.

So we need to compute partial derivatives of $f(x,y) = x^2 - y^2$.

$f_x(x,y) = 2x$ and $f_y(x,y) = -2y$. Hence at (x_0, y_0, z_0) we have that $(-2x_0, -2y_0, 1)$ is a normal vector.

We need to find x_0, y_0 such that

$$(-2x_0, -2y_0, 1) = c(3,1,2)$$

for some c. Comparing the 3^{rd} components, we get $c = 1/2$.

Therefore $-2x_0 = 3/2$ and $-2y_0 = 1$, which implies

$x_0 = \frac{-3}{4}$, $y_0 = \frac{1}{4}$. And so $z_0 = \left(\frac{-3}{4}\right)^2 - \left(\frac{1}{4}\right)^2 = \frac{8}{16} = \frac{1}{2}$.

Lecture 13: Differentiability and affine approximation

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How can we make sure that a tangent plane exists?

We saw that, if tangent plane exists, then its equation is

$$z = f(x_0,y_0) + f_x(x_0,y_0)\,(x-x_0) + f_y(x_0,y_0).$$

Therefore it is graph of the affine function (degree 1)

$$\boxed{L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)\,(x-x_0) + f_y(x_0,y_0)\,(y-y_0)}$$

Existence of tangent plane is equivalent to saying that $L(x,y)$ is "<u>fairly good</u>" approximation of $f(x,y)$ for (x,y)'s that are close to (x_0,y_0). Here is what we mean by "<u>fairly good</u>".

<u>Definition</u>. f is called <u>differentiable</u> at (x_0,y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)-L(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0.$$

This means "the error term" $f(x,y)-L(x,y)$ goes to zero much faster than the distance between (x,y) and (x_0,y_0). And

f is differentiable at (x_0,y_0) exactly when $z=f(x,y)$ has a tangent plane at $(x_0,y_0,f(x_0,y_0))$.