Lecture 14: Differentiation

Monday, October 24, 2016 8:40 AM

In the previous lecture we defined a differentiable function:

f is differentiable at (x_0,y_0) if $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)-L(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}=0$

where $L(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$.

As we have seen before, it is NOT very easy to deal with limits of multi-variable functions. Lucky us the following theorem can help us to get differentiability without dealing with limits!

Theorem If in a disk around a point x_0 all the partial derivatives are continuous, then f is differentiable at x_0.

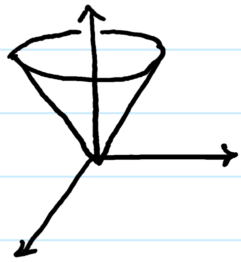
Ex. Find all the points where $f(x,y)=\sqrt{x^2+y^2}$ is differentiable.

Solution. (As we have seen before,) $f_x(x,y)=\dfrac{x}{\sqrt{x^2+y^2}}$ and $f_y(x,y)=\dfrac{y}{\sqrt{x^2+y^2}}$. So f_x and f_y are continuous everywhere except possibly at $(0,0)$. Hence for any $(x_0,y_0)\neq(0,0)$, f has continuous partial derivative in a disk centered at (x_0,y_0) with radius smaller than $\sqrt{x_0^2+y_0^2}$, e.g. $\frac{1}{2}\sqrt{x_0^2+y_0^2}$. Therefore the above theorem implies that f is differentiable at (x_0,y_0). Next we notice

Lecture 14: Differentiability, approximation

Monday, October 24, 2016 8:57 AM

$f(x,0)=\sqrt{x^2}=|x|$ is not differentiable at $x=0$ as a single-variable function. So f_x does NOT exist at $(0,0)$, which implies f is not differentiable at $(0,0)$.



Remark. $z=\sqrt{x^2+y^2}$ is equation of a cone which clearly does NOT have a tangent plane at $(0,0,0)$.

Ex. Find equation of the tangent plane at $(1,1,\sqrt{2})$ in the above example.

Solution.
$$z=\sqrt{2}+f_x(1,1)(x-1)+f_y(1,1)(y-1)$$
$$=\sqrt{2}+\frac{1}{\sqrt{2}}(x-1)+\frac{1}{\sqrt{2}}(y-1)$$
$$z=\frac{\sqrt{2}}{2}x+\frac{\sqrt{2}}{2}y.$$

Ex. Suppose $f(x,y)=\frac{1}{\pi}\cos\left(\frac{\pi}{2}x^2y\right)$. Find linear approximation of $f(-1.1,1.2)$.

Solution. $f(x,y)\approx f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$

$f_x(x,y)=-xy\sin\left(\frac{\pi}{2}x^2y\right)$ and $f_y(x,y)=\frac{-x^2}{2}\sin\left(\frac{\pi}{2}x^2y\right)$

$f(-1,1)=\frac{1}{\pi}\cos\left(\frac{\pi}{2}\right)=0,$

$f_x(-1,1)=\sin\left(\frac{\pi}{2}\right)=1$ and $f_y(-1,1)=\frac{-1}{2}\sin\left(\frac{\pi}{2}\right)=\frac{-1}{2}$

Lecture 14: Approximation

Monday, October 24, 2016 9:10 AM

So $f(x,y) \approx (x+1) - \frac{1}{2}(y-1)$, which implies

$$f(-1.1, 1.2) \approx (-0.1) - \frac{1}{2}(0.2) = -0.2.$$