Lecture 16: Parametrization of curves

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In the previous lecture we studied single-variable vector-valued functions. We saw:

$$\boxed{\lim_{t\to t_0} \vec{r}(t) = \left(\lim_{t\to t_0} x(t), \lim_{t\to t_0} y(t), \lim_{t\to t_0} z(t)\right)}$$

$$\boxed{\vec{r}\,'(t) = (x'(t), y'(t), z'(t))}$$

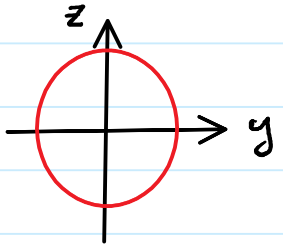
> Parametrization of tangent line of $\vec{r}(t)$ at $\vec{r}(t_0)$
>
> is given by $\quad \ell(t) = \vec{r}(t_0) + t\,\vec{r}\,'(t_0)$

Ex. Parametrize the circle of radius 2, centered at the origin, in the yz-plane.

Solution. Since it is in the yz-plane, $x=0$.

$y = \cos\theta$, $z = \sin\theta$ for $0 \le \theta \le 2\pi$.

So we get $\vec{r}(\theta) = (0, \cos\theta, \sin\theta)$.



Ex. Parametrize the curve of intersection of

$$x^2+y^2=4 \quad \text{and} \quad x+y+z=1$$

Solution. Whenever you see an equation of the form,
$A^2+B^2=R^2$ you should think of $A = R\cos\theta$, $B = R\sin\theta$!

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So $x = 2\cos\theta$, $y = 2\sin\theta$ for $0 \leq \theta \leq 2\pi$,

and $z = 1 - x - y = 1 - 2\cos\theta - 2\sin\theta$. Therefore

$$\vec{r}(\theta) = (2\cos\theta,\ 2\sin\theta,\ 1 - 2\cos\theta - 2\sin\theta).$$

Ex. Parametrize the curve of intersection of

$x^2 + z^2 = 1$, $y^2 + z^2 = 1$, and $x \geq 0$.

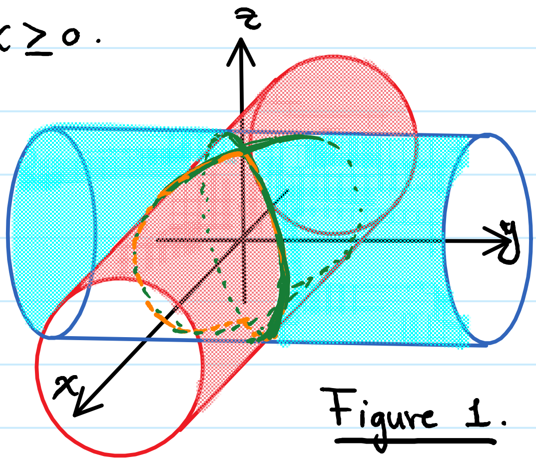

Figure 1.

Solution. In these figures, the orange curve is what we are looking for. As you can see in Figure 2., for a given x, z, there are two possible values for y. But having y, z, there is a unique x. (For points

From above it looks like

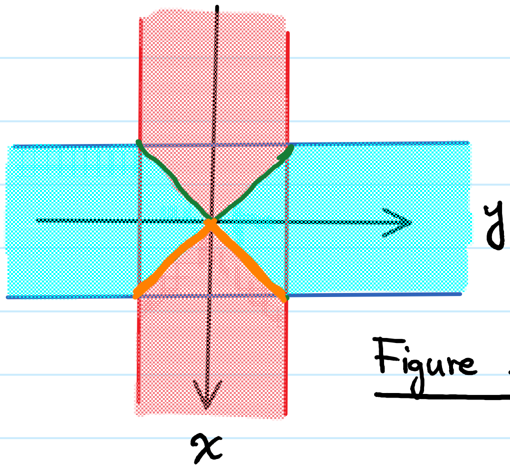

Figure 2.

From fron (through x-axis):

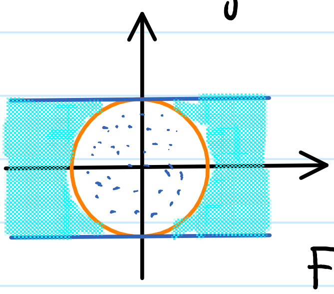

Figure 4.

From left (through y-axis):

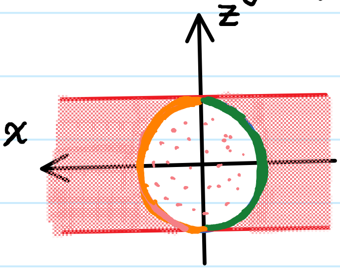

Figure 3.

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on the orange curve.) So we start with y, z components: $y^2+z^2=1$. So let $y=\cos\theta$, $z=\sin\theta$ for $0\le\theta\le 2\pi$ (as we can see in Figure 4, we have the full circle.)

Since $x^2+z^2=1$, we get

$$x^2=1-z^2=1-\sin^2\theta=\cos^2\theta.$$

On the other hand, $x\ge 0$. Hence $x=|\cos\theta|$.

Overall we get $\vec{r}(\theta)=(|\cos\theta|, \cos\theta, \sin\theta)$ for $0\le\theta\le 2\pi$.

Ex. How can we visualize the curve $\vec{r}(t)=(\cos t, \sin t, t)$?

To visualize a curve given by a parametrization, one can try to find relations between its components. This way we might be able to find surfaces that contain the given curve.

Solution. We observe that $x=\cos\theta$, $y=\sin\theta$ satisfy $x^2+y^2=1$. Hence this curve is part of this cylinder.

Lecture 16: Parametrization and chain rule

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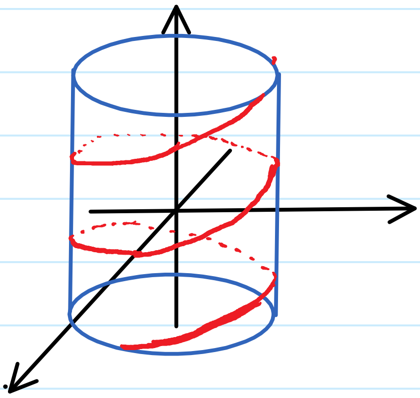
(Think about it as a bumblebee which is at $(\cos t, \sin t, t)$ after t seconds. So it is at $(1,0,0)$ at $t=0$, and it flies upward, but its shadow on ground just rotates on a circle centered at the origin.) [It is called a helix.]



Now suppose in the above example, the temperature at any point is given $T(x,y,z)$.

We'd like to know what is the rate of change of temperture as the bumblebee flies away? More generally:

For a given vector-valued function $\vec{r}(t) = (x(t), y(t))$ (it might have three components) and a given two-variable function $f(x,y)$, how can we compute $\frac{d}{dt} f(\vec{r}(t))$?

By definition,

$$\frac{d}{dt} f(\vec{r}(t)) = \lim_{\Delta t \to 0} \frac{f(x(t+\Delta t), y(t+\Delta t)) - f(x(t), y(t))}{\Delta t}$$

To understand this limit, we approximate $f(x,y)$ by an affine function for (x,y) close to $(x(t), y(t))$.

Lecture 16: Chain rule

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We know that for any (a,b) we have

$$f(x,y) \approx f(a,b) + f_x(a,b)\,(x-a) + f_y(a,b)\,(y-b)$$

if f is differentiable at (a,b) and (x,y) is close to (a,b).

Using this for $(a,b) = (x(t), y(t))$ and

$$(x,y) = (x(t+\Delta t),\, y(t+\Delta t)),$$

we get:

$$f(x(t+\Delta t), y(t+\Delta t)) \approx$$

$$f(x(t),y(t)) + f_x(x(t),y(t))\,(x(t+\Delta t) - x(t)) + f_y(x(t),y(t))\,(y(t+\Delta t) - y(t))$$

Hence $$\frac{f(x(t+\Delta t), y(t+\Delta t)) - f(x(t),y(t))}{\Delta t} \approx$$

$$f_x(x(t),y(t))\,\frac{x(t+\Delta t)-x(t)}{\Delta t} + f_y(x(t),y(t))\,\frac{y(t+\Delta t)-y(t)}{\Delta t}$$

As $\Delta t \to 0$, we get

$$\frac{x(t+\Delta t)-x(t)}{\Delta t} \to x'(t) \quad \text{and} \quad \frac{y(t+\Delta t)-y(t)}{\Delta t} \to y'(t)$$

So we get

Chain Rule. $$\frac{d}{dt} f(\vec{r}(t)) = \frac{\partial f}{\partial x}(\vec{r}(t)) \cdot \frac{dx}{dt}(t) + \frac{\partial f}{\partial y}(\vec{r}(t)) \cdot \frac{dy}{dt}(t)$$

Sometimes we write: $$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$