Lecture 19: Properties of directional derivatives

Friday, November 4, 2016 8:28 AM

In the previous lecture we got the following formulas for directional derivatives (using chain rule): for any point P_0

$$\boxed{\begin{array}{l} D_{\vec{u}} f(P_0) = \vec{\nabla} f(P_0) \cdot \vec{u} = \|\vec{\nabla} f(P_0)\| \cos\theta \\ \text{where } \theta \text{ is the angle between } \vec{\nabla} f(P_0) \text{ and } \vec{u} \end{array}}$$

<u>Ex.</u> • Find the maximum rate of increase of

$$f(x,y) = x e^{x+y^2}$$

at $(-1,1)$. In which direction does it occur?

• What is the directional derivative of f in the direction of $\vec{v} = (1,1)$?

• Does f increase in the direction of $\vec{v}$? How about $-\vec{v}$?

<u>Solution</u>. The max. rate of increase at P_0 is $\|\nabla f(P_0)\|$ and it occurs in the direction of $\nabla f(P_0)$.

$$\nabla f(x,y) = \left(e^{x+y^2} + x e^{x+y^2},\ 2yx\, e^{x+y^2}\right)$$

So $\nabla f(-1,1) = (e^2 - e^2, -2e^2) = (0, -2)$

Hence max. rate of change of f at $(-1,1) = \|\nabla f(-1,1)\|$
$= 2$.

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. First we need to normalize $\vec{v} = (1,1)$:

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

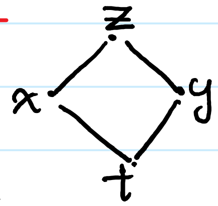
$$D_{\vec{u}} f(-1,1) = \nabla f(-1,1) \cdot \vec{u} = (0,-2)\cdot\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\sqrt{2}.$$

. f decreases in the direction of $\vec{v}$ as $\underline{D_{\vec{u}} f(-1,1)}$ is <u>negative</u> .

Since $D_{-\vec{u}} f(-1,1) = -D_{\vec{u}} f(-1,1) = \sqrt{2} > 0$,

f increases in the direction of $-\vec{v}$. ■

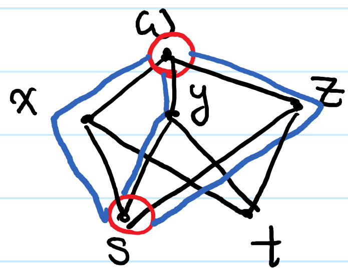
We have seen chain rule in the setting



where z depends on certain variables, say x and y; and x, y depend on t.

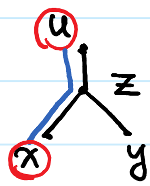
A similar formula works in much more generality:

Say we have the following diagram of dependence of variables.



Then $$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

or $$\frac{\partial u}{\partial x} = \frac{du}{dz}\frac{\partial z}{\partial x}$$



Lecture 19: Chain rule; Implicit differentiation

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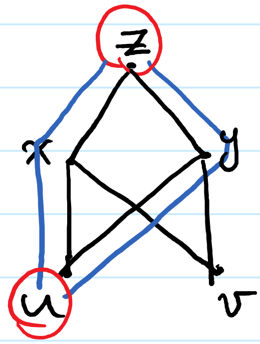
Ex. Suppose $z = \cos(x-y)$, $x = 2u - v$ $y = u + 2v$.

Find $\frac{\partial z}{\partial u}$.



Solution.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial x} = -\sin(x-y), \quad \frac{\partial x}{\partial u} = 2, \quad \frac{\partial z}{\partial y} = \sin(x-y), \quad \frac{\partial y}{\partial u} = 1.$$

So

$$\frac{\partial z}{\partial u} = (-\sin(x-y))(2) + (\sin(x-y))(1)$$

$$= -\sin(x-y)$$

$$= -\sin((2u-v)-(u+2v))$$

$$= -\sin(u-3v).$$

we are NOT done here! we should write our answer in terms of u and v.

unless you are explicitly told it is OK to write our answer in terms of all variables.

- Suppose we are told y is a function of x and $f(x,y) = 0$. Then $(x, y(x))$ is (part of) a level curve.

So $\nabla f(x,y) \cdot (1, \frac{dy}{dx}) = 0$. Hence

$(f_x(x,y), f_y(x,y)) \cdot (1, \frac{dy}{dx}) = 0$, which implies

$f_x(x,y) + \frac{dy}{dx} f_y(x,y) = 0$. So $\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$

Lecture 19: Implicit differentiation

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By a similar argument, if z is a function of x and y, and $F(x,y,z)=0$ for some function F, Then we have

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}$$

<u>Ex.</u> Suppose y is defined implicitly by

$$xy + \cos(y) - e^{xy} = 0$$

Find $\frac{dy}{dx}$ in terms of x and y.

<u>Solution.</u> ① Let $f(x,y) = xy + \cos(y) - e^{xy}$. By the formula we know:

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{y - y e^{xy}}{x - \sin y - x e^{xy}}$$

② We differentiate both sides of $xy + \cos(y) - e^{xy} = 0$ in terms of x and view y as a function of x:

$$\frac{d}{dx}(xy) = y + x\frac{dy}{dx}, \qquad \frac{d}{dx}\cos(y) = -\sin(y)\frac{dy}{dx},$$

$$\frac{d}{dx}\left(e^{xy}\right) = \left(y + x\frac{dy}{dx}\right)e^{xy}. \quad \text{So}$$

$$\underbrace{y}_{} + \underbrace{x\frac{dy}{dx}}_{} - \underbrace{\sin(y)\frac{dy}{dx}}_{} - \underbrace{y e^{xy}}_{} - \underbrace{x e^{xy}\frac{dy}{dx}}_{} = 0$$

Lecture 19: Implicit differentiation

Sunday, November 6, 2016 10:18 PM

Solving for $\frac{dy}{dx}$ we get the same answer.

"Implicit differentiation is particularly useful in problems in physics where we know certain quantities satisfy certain equation and we would like to know how fast one of them changes as we vary the other one."

This part is for interested students: here we show why the implicit differentiation is true in 3-variable setting.

Suppose z is a function of x and y. And for a three variable function F, we have $F(x,y,z)=0$. So graph of $z=z(x,y)$ is part of the level surface of $F(x,y,z)=0$. Hence $\nabla F(x,y,z)$ is a normal vector of the tangent plane. On the other hand, $\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)$ is a normal vector of the tangent plane of graph $z=z(x,y)$ of z is. Therefore for some c, we have $\nabla F = c\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)$. Thus $\frac{\partial F}{\partial z} = -c$, $\frac{\partial F}{\partial x} = c\frac{\partial z}{\partial x}$, $\frac{\partial F}{\partial y} = c\frac{\partial z}{\partial y}$. So $\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}$ and $\frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}$.