Lecture 20: Higher order differentiation

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To understand single-variable functions better higher ordered derivatives are used (as you have seen before). The same is true for multi-variable functions. It is particularly important in dealing with optimization problems.

What are higher-order derivatives?

Let $f(x,y) = x^2 + x^2y + y^3$. Then

$$f_x(x,y) = 2x + 2xy \quad \text{and} \quad f_y(x,y) = x^2 + 3y^2.$$

Now we can talk about partial derivatives of f_x and f_y in terms of x and y:

Partial derivative of f_x with respect to $x = f_{xx}(x,y) = 2+2y$

Partial derivative of f_x with respect to $y = f_{xy}(x,y) = 2x$

Partial derivative of f_y with respect to $x = f_{yx}(x,y) = 2x$

Partial derivative of f_y with respect to $y = f_{yy}(x,y) = 6y$

The 2^{nd} partial derivatives, (also called iterated partial derivatives) are written: $f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$, $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \, \partial x}$

$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \, \partial y}$, $f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$.

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Ex. Find 2^{nd}-order partial derivatives of $f(x,y) = \log(x^2+y)$.

Solution. $f_x = \dfrac{2x}{x^2+y}$, $f_y = \dfrac{1}{x^2+y}$

So

$$f_{xx} = \frac{2(x^2+y) - (2x)(2x)}{(x^2+y)^2} = \frac{2y - 2x^2}{(x^2+y)^2}.$$

$$f_{xy} = \frac{-2x}{(x^2+y)^2}, \quad f_{yx} = \frac{-2x}{(x^2+y)^2}, \quad f_{yy} = \frac{-1}{(x^2+y)^2}.$$

In the above example you see $f_{xy} = f_{yx}$. This is true in a fairly general setting: if all the 2^{nd} order derivatives are continuous. So for nice function it is better to choose a good order.

Ex. Let $g(x,y) = e^{e^{(x^2+\ln x)}} + xy$. Find g_{xy}.

Solution As you can see, a part of g is a complicated function of x. So it takes time to compute g_x, and then g_{xy}. But since it is a nice function, $g_{xy} = g_{yx}$. And $g_y = x$. So $g_{xy} = g_{yx} = 1$.

We can talk about iterated partial derivatives of more than

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2 variable functions and the same technique can be used:

Ex. Let $g(x,y,z) = \ln(\cos(x^2+y^2)) + xy^2z^3$. Find g_{xyz}.

Solution. Since g is a nice function, we can choose any order we like to find g_{xyz}.

g is easiest as a function of z. So let's compute g_z first: $g_z = 3xy^2z^2$.

Now it is easy enough to work with any order of x and y: $g_{zx} = 3y^2z^2$ and $g_{zxy} = 6yz^2$.

Hence $g_{xyz} = g_{zxy} = 6yz^2$.

Optimization problems: finding maximum and minimum of a multi-variable function.

As we have seen before, if directional derivative of f in the direction of $\vec{u}$ is positive, then f increases in the direction of $\vec{u}$. And we have seen the directional derivative of f in the direction of $\vec{\nabla} f(p_0)$ is $\|\vec{\nabla} f(p_0)\|$. So if f has a local maximum at p_0, then $\vec{\nabla} f(p_0) = \vec{0}$.

Lecture 20: Local max and min, critical points

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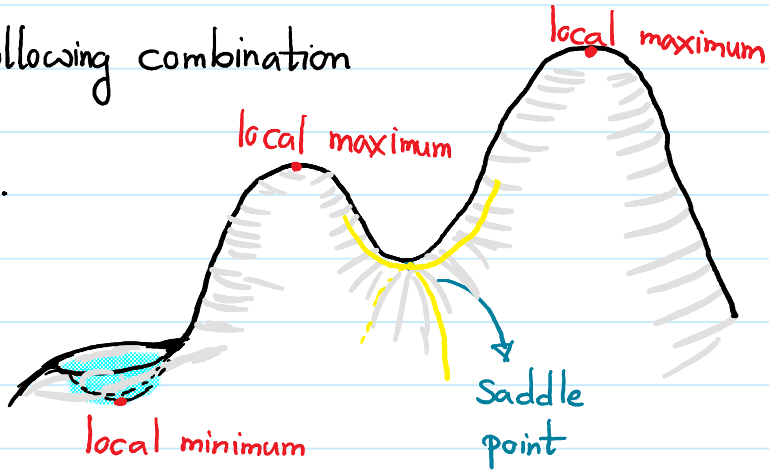
Similarly we have $D_{\vec{u}} f(p_0) < 0$ implies f decreases in the direction of $\vec{u}$. And directional derivative in the direction of $\overrightarrow{\nabla f}(p_0)$ is $-\|\nabla f(p_0)\|$. So we get

$$\boxed{\text{If } f \text{ is differentiable at } p_0 \text{ and } f \text{ has a local maximum or a local minimum at } p_0, \text{ then } \overrightarrow{\nabla f}(p_0) = \vec{0}}$$

For instance consider the following combination of "mountains" and "lake".



Because of the above important box we define a critical point of f as follows:

> **Definition**. We say p_0 is a critical point of f if either f is NOT differentiable at p_0, or $\overrightarrow{\nabla f}(p_0) = 0$.

So we have

> If f has a local max or a local min at p_0, then p_0 is a critical point of f.

Lecture 20: Saddle point

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- Not all the critical points are going to give us local extreme values.

- If f does NOT have a local max or a local min at a critical point P_0, then P_0 is called a saddle point.