Lecture 21: Critical points

Wednesday, November 9, 2016 8:46 AM

In the previous lecture we defined the notion of critical point.

We said p_0 is a critical point of f exactly when either f is NOT differentiable at p_0 or $\vec{\nabla} f(p_0) = \vec{0}$.

Ex. Find all the critical points of

$$f(x,y) = x^3 + y^4 - 6x - 2y^2.$$

Solution. f is everywhere differentiable. So we need to solve $\vec{\nabla} f(x,y) = \vec{0}$.

$$\vec{\nabla} f(x,y) = (3x^2 - 6,\ 4y^3 - 4y) = (0,0)$$

So $\begin{cases} 3x^2 - 6 = 0 \\ 4y^3 - 4y = 0 \end{cases} \Rightarrow \begin{cases} x = \pm\sqrt{2} \\ y = 0, -1, \text{ or } 1. \end{cases}$

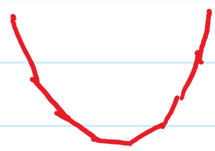
So f has 6 critical points $(-\sqrt{2}, -1)$, $(-\sqrt{2}, 0)$, $(-\sqrt{2}, 1)$, $(\sqrt{2}, -1)$, $(\sqrt{2}, 0)$, and $(\sqrt{2}, 1)$.

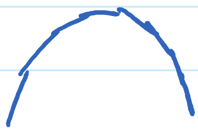
How can we determine if f has a local extremum at a critical point p_0?

In single-variable case, to find local extremum of $f(x)$

Lecture 21: Local extremum; Second derivative test

Wednesday, November 9, 2016 9:01 AM

you found its critical points by solving $f'(x)=0$. Then you could use $f''(x_0)$. If $f''(x_0)>0$, then f' is increasing so graph of your function would look like , which implies f has a local min at x_0.

If $f''(x_0)<0$, then f' is decreasing so the slope of tangent lines are decreasing. Thus graph of function looks like , which implies f has a local maximum at x_0. If $f''(x_0)=0$, you cannot conclude anything!

For multi-variable functions, again we can use second derivatives. This time, however, we have to use all the 2^{nd} order partial derivatives:

<u>Two-variable case</u> Consider the matrix

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}.$$

It is called the hessian matrix of f. The following table can help us to determine if a critical point gives us a local maximum, a local minimum or it is a saddle point.

Lecture 21: The second order derivative test

Friday, November 11, 2016 4:49 PM

When all the second order derivatives are continuous, then the sign of f_{xx} and the determinant of the hessian matrix

$$D = f_{xx}\cdot f_{yy} - f_{xy}\cdot f_{yx} = f_{xx}\cdot f_{yy} - f_{xy}^2$$

can help us to classify critical points:

critical points	$f_{xx}\cdot f_{yy} - f_{xy}^2$	f_{xx}	result
	$+$	$+$	local minimum
	$+$	$-$	local maximum
	$-$	$\times$	saddle point
	0		inconclusive

it does NOT matter

For a three-variable function, we can use the hessian matrix again:

$$\begin{bmatrix} f_{xx} & f_{yx} & f_{zx} \\ f_{xy} & f_{yy} & f_{zy} \\ f_{xz} & f_{yz} & f_{zz} \end{bmatrix}.$$

Let $D_1 = f_{xx}$, $D_2 = \det\begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy}\end{bmatrix}$,

and $D_3 = \det\begin{bmatrix} f_{xx} & f_{yx} & f_{zx} \\ f_{xy} & f_{yy} & f_{zy} \\ f_{xz} & f_{yz} & f_{zz} \end{bmatrix}$.

(0) If one of D_1, D_2, or D_3 is equal to 0, then the test is inconclusive.

(1) If $D_1 > 0$, $D_2 > 0$, $D_3 > 0$, then f has a local minimum.

(2) If $D_1 < 0$, $D_2 > 0$, $D_3 < 0$, then f has a local maximum.

(3) If all D_i's are non-zero but (1) and (2) do not happen, it is a saddle point.

Lecture 21: The second order derivative test

Friday, November 11, 2016 5:44 PM

Ex. Find all local maximum, local minimum and saddle points of

$$f(x,y) = x^3 + y^4 - 6x - 2y^2.$$

Solution. We have already found all of f's critical points:

$$(-\sqrt{2},-1), (-\sqrt{2},0), (-\sqrt{2},1), (\sqrt{2},-1), (\sqrt{2},0), (\sqrt{2},1).$$

Since f is a polynomial, all of its 2^{nd} ordered partial derivatives are continuous. So we can use the 2^{nd} order derivative test. Hence we need to compute all the 2^{nd} order derivatives. We start with f_x and f_y. We have $f_x(x,y) = 3x^2 - 6$ and $f_y(x,y) = 4y^3 - 4y$.

So $f_{xx} = 6x$, $f_{xy} = f_{yx} = 0$, $f_{yy} = 12y^2 - 4$. Therefore

$$D = f_{xx} \cdot f_{yy} - f_{xy}^2 = (6x)(12y^2-4) = 24x(3y^2-1)$$

Critical points	$D = 24x(3y^2-1)$	$f_{xx} = 6x$	Conclusion
$(-\sqrt{2},-1)$	$-$	~~~	saddle point
$(-\sqrt{2},0)$	$+$	$-$	local maximum
$(-\sqrt{2},1)$	$-$	~~~	saddle point
$(\sqrt{2},-1)$	$+$	$+$	local minimum
$(\sqrt{2},0)$	$-$	~~~	Saddle point
$(\sqrt{2},1)$	$+$	$+$	local minimum

Lecture 21: bounded and closed regions

Friday, November 11, 2016 7:22 PM

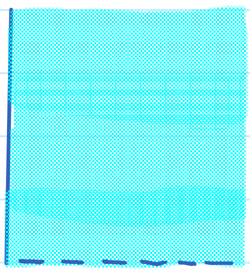
Next we would like to study global maximum and global minimum.

First, we discuss certain conditions which guarantee the existence of global max and global min. Then, we will see how we can find these values.

Theorem. Any continuous function on a bounded and closed region of (line or) plane (or space) has a global max and global min.

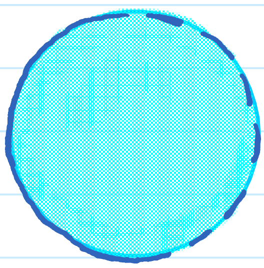
So we need to understand what a bounded region is.

- A region is called bounded if it can be contained within a disk.



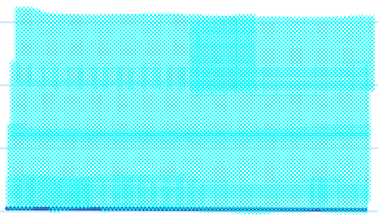
(The positive corner of plane including the positive part of y-axis and excluding the positive part of the x-axis.)

Unbounded



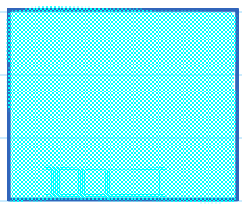
(Inside a disk and parts of the surrounding circle (its boundary).)

Bounded



(The upper-half plane, including the x-axis)

Unbounded



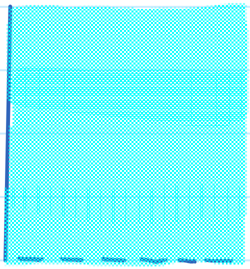
(The region enclosed by a rectangle; including the rectangle)

Bounded

Lecture 21: Closed region

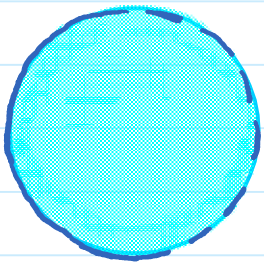
Friday, November 11, 2016 7:53 PM

A region is called closed if it contains its boundary. Alternatively: if with points in this region you get closer and closer to a point P_0, then P_0 is a point of this region, too. For example:



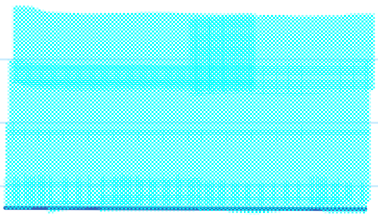
(The positive corner of plane including the positive part of y-axis and excluding the positive part of the x-axis.)

Unbounded, Not closed



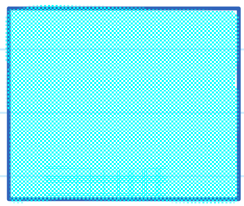
(Inside a disk and parts of the surrounding circle (its boundary).)

Bounded, Not closed



(The upper-half plane, including the x-axis)

Unbounded, closed



(The region enclosed by a rectangle; including the rectangle)

Bounded, closed