Lecture 24: Review of Lagrange multiplier method

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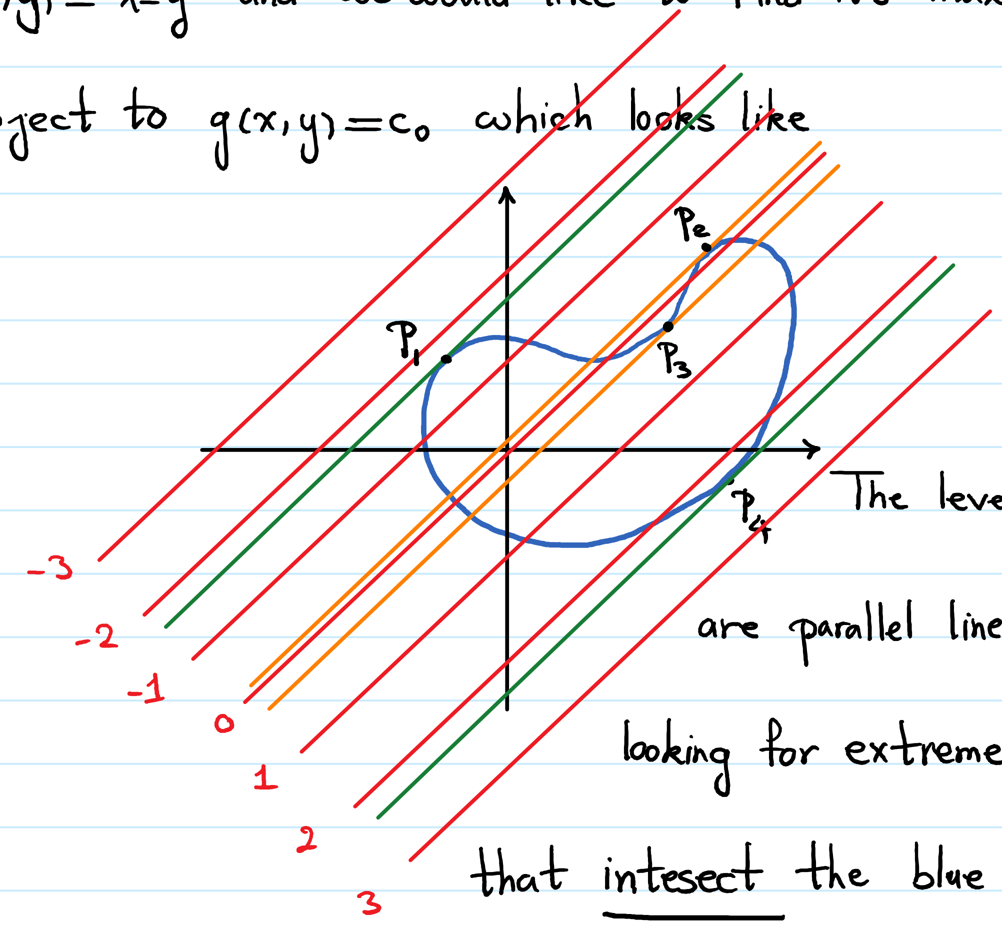
Lagrange Multiplier Method. Suppose f has continuous partial derivatives. If f has a max. or a min. at p_0 subject to $g(x,y)=c_0$, then $\nabla f(p_0) = c\,\nabla g(p_0)$ for some c.

Let's try to understand what this method gives. Suppose $f(x,y) = x-y$ and we would like to find its max. and min. subject to $g(x,y)=c_0$ which looks like



The level curves of f are parallel lines, and we are looking for extreme such lines that intesect the blue curve. Since these are supposed to be extreme intersecting lines, they should be tangent. (The green lines). Lagrange multiplier method gives

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gives us all the points where the level curve of f is tangent to the level curve of g. Notice that ∇f is perpendicular to its level curve, ∇g is perpendicular to its level curve; so the level curves are tangent exactly when ∇f is parallel to ∇g.

Now notice that in the above example, there are two other tangent level curves (the orange lines). So the Lagrange multiplier method gives us four points.

Now comparing $f(p_1)$, $f(p_2)$, $f(p_3)$, and $f(p_4)$ one can see that $f(p_1)$ is min. and $f(p_4)$ is max.

- Here is an example similar to what you saw in the previous lecture. But here we address only the concepts behind it:

Find the max. and the min. of $x+y$ subject to $xy=1$ (if they exist). (See the next page for the picture)

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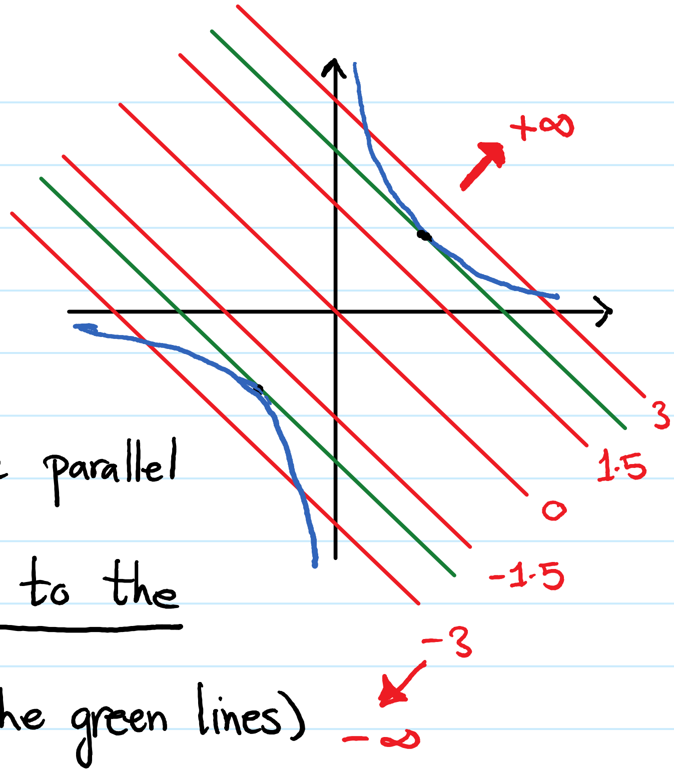

The same idea as before, without doing any computation, we know that Lagrange multi. method gives us lines that are parallel to red lines and are tangent to the blue curve. (which means the green lines)

On the other hand, as it can be seen in the picture, f, on the blue curve, has no max and no min.

No, if we add another assumption: $x, y \geq 0$, then the point given by Lagrange multiplier method gives us a min. for f. And still there is no max.

> When $g(x,y)=c_0$ is NOT bounded, one has to understand how f behaves as we go to infinity along $g(x,y)=c_0$.

Ex. Find max. and min. of x^2y subject to $2x^2+y^2=1$.

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Solution. Let $g(x,y) = 2x^2+y^2$. Since $g(x,y)=1$ is an ellipse, which is closed and bounded, and f is continuous, f has a max. and a min. on this ellipse.

By Lagrange multiplier method, if f has a max. or a min at p_0, then

$$\begin{cases} \nabla f(p_0) = c\,\nabla g(p_0) \\ g(p_0) = 1. \end{cases} \Rightarrow \begin{cases} (2xy, x^2) = c\,(4x, 2y) \\ 2x^2+y^2=1. \end{cases}$$

$$\Rightarrow \begin{cases} ① \; 2xy = 4cx \Rightarrow \text{either } x=0 \text{ or } y=2c \\ ② \; x^2 = 2cy \\ ③ \; 2x^2+y^2 = 1 \end{cases}$$

Case 1. $x \neq 0$.

Then $y = 2c$ and $x^2 = (2c)^2$. So, by ③,

$2(2c)^2 + (2c)^2 = 1$. Therefore $c^2 = \frac{1}{12}$; thus $c = \pm\frac{1}{\sqrt{12}}$.

In this case $f(x,y) = x^2 y = (2c)^2(2c) = 8c^3 = \pm\frac{8}{12\sqrt{12}}$.

$$= \pm\frac{\sqrt{3}}{9}.$$

Case 2. $x=0$. In this case $f(x,y)=0$.

Comparing the possible values of f, we have that max. $= \frac{\sqrt{3}}{9}$ and min. $= -\sqrt{3}/9$.

Lecture 24: Vector valued functions reviewed

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Let me quickly recall that, if a moving particle is at $\vec{r}(t)$, then

$$\boxed{\begin{array}{ll} \text{velocity}: & \vec{v}(t) = \vec{r}\,'(t). \\ \text{acceleration}: & \vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t). \\ \text{speed}: & s(t) = \|\vec{v}(t)\| = \|\vec{r}\,'(t)\|. \end{array}}$$

Ex. Suppose a particle is at $((t-1)^2, (t-1)^3)$ at time t.

(a) Sketch its path for $0 \le t \le 2$.

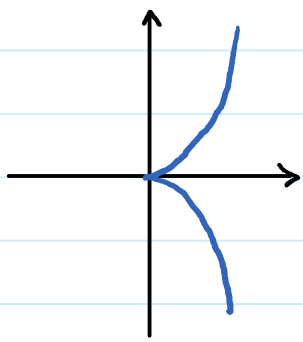
(b) Find its velocity, speed, and acceleration.

Solution (a) Since $x = (t-1)^2$ and $y = (t-1)^3$,

for $t \ge 1$, $y = x^{3/2}$, $(x \ge 0)$

for $t < 1$, $y = -x^{3/2}$, $(x \ge 0)$.

Hence its path looks like:



(b) Velocity: $\vec{v}(t) = \vec{r}\,'(t) = (2(t-1), 3(t-1)^2)$.

acceleration: $\vec{a}(t) = \vec{v}\,'(t) = (2, 6(t-1))$.

speed: $s(t) = \|\vec{v}(t)\| = \sqrt{4(t-1)^2 + 9(t-1)^4}$

$$= |t-1|\sqrt{4+9(t-1)^2} = |t-1|\sqrt{9t^2-18t+13}.$$

Lecture 24: Regular curve

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In the previous example, we notice there is a sharp point at $(0,0)$. Particle reaches to that point at $t=1$. At this time, its velocity is $(0,0)$. In order for the particle to change his direction of movement completely, it has to stop, which means a curve which is parametrize by $\vec{r}(t)$ has a sharp point at time t_0 only when $\vec{r}\,'(t_0)=0$. That is why we define a point to be regular if $\vec{r}\,'(t_0)\neq 0$.