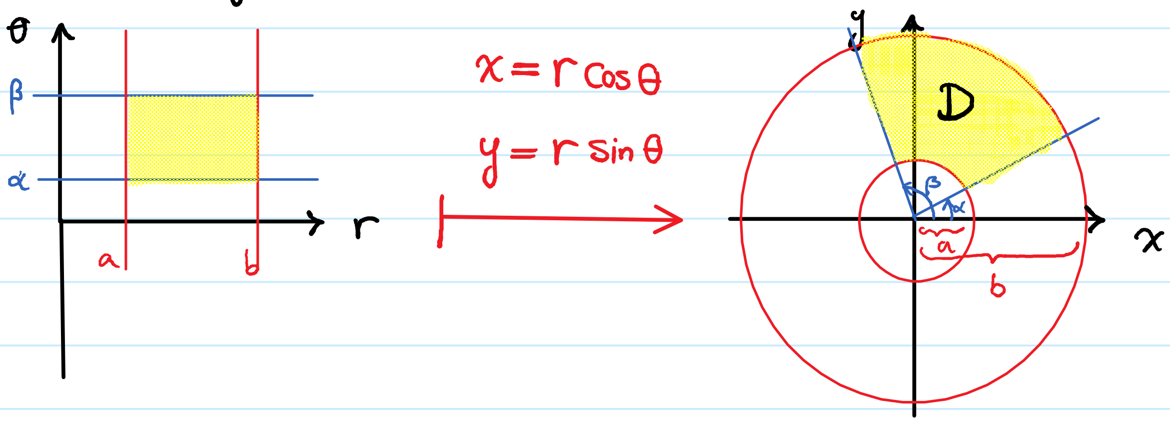
Lecture 29: Polar coordinates and double integrals

Friday, December 2, 2016 8:27 AM

To understand the connection between polar coordinates and double integrals, we start with a "rectangle" in polar coordinates:



What is $\iint_D dA$? For any region D, $\iint_D dA = \text{area}(D)$.

$$\text{Area}(D) = \frac{\beta-\alpha}{2\pi}\left(\pi b^2 - \pi a^2\right) = (\beta-\alpha)\left(\tfrac{1}{2}b^2 - \tfrac{1}{2}a^2\right)$$

$$= \left(\int_\alpha^\beta d\theta\right)\left(\int_a^b r\,dr\right) = \int_\alpha^\beta \int_a^b r\,dr\,d\theta$$

In fact this is a general rule:

$$\boxed{\begin{array}{l} D:\ \text{in polar coordinates } \alpha \le \theta \le \beta,\ a \le r \le b, \\ \displaystyle\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta. \end{array}}$$

Remark. To remember the formula, you can think about it as $dx\,dy = r\,dr\,d\theta$. Do NOT forget the r factor.

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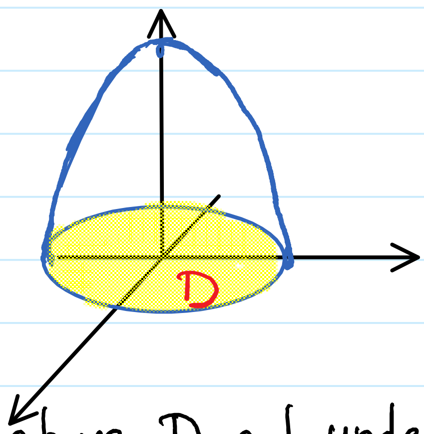
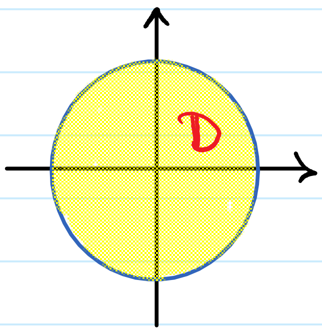
Ex. Find volume of the solid enclosed by $z = 4 - x^2 - y^2$ and $z = 0$.



Solution. We sketch $z = 4 - x^2 - y^2$ and find its intersection with $z = 0$. So the considered solid is above D and under graph $z = 4 - x^2 - y^2$, where D is the disk inside $x^2 + y^2 = 4$.

So volume $= \iint_D 4 - x^2 - y^2 \, dA$.



In polar coordinates D has a simple description: $0 \le \theta \le 2\pi$ (always start with θ)

$0 \le r \le 2$

So
$$\iint_D 4 - x^2 - y^2 \, dA = \int_0^{2\pi} \int_0^2 \left(4 - r^2\cos^2\theta - r^2\sin^2\theta\right) r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 (4 - r^2) r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left(4\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_{r=0}^{r=2} d\theta = \int_0^{2\pi} (2)\, 2^2 - \frac{2^4}{4} \, d\theta$$

$$= \int_0^{2\pi} 4 \, d\theta = 4\theta \Big|_0^{2\pi} = 8\pi.$$

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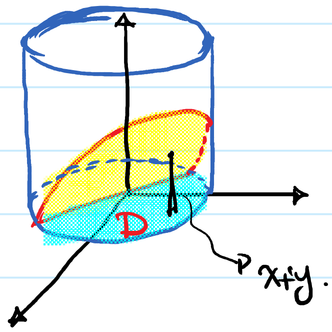
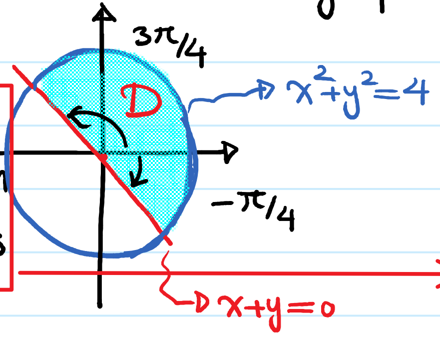
Ex. Find volume of the solid in cylinder $x^2+y^2=4$, under plane $x+y=z$, and above the xy-plane.

Solution. First we have to find the projection of the solid to the xy-plane.



Now D has an easy description in polar coordinates



$$-\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$$
$$0 \le r \le 2$$

$$\text{volume} = \iint_D x+y \, dA = \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^2 (r\cos\theta + r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_{-\pi/4}^{3\pi/4} (\cos\theta + \sin\theta) \left.\frac{r^3}{3}\right|_0^2 d\theta$$

$$= \frac{8}{3}(\sin\theta - \cos\theta) \Big|_{-\pi/4}^{3\pi/4}$$

$$= \frac{8}{3}\left[\left(\sin\frac{3\pi}{4} - \cos\frac{3\pi}{4}\right) - \left(\sin\left(-\frac{\pi}{4}\right) - \cos\left(-\frac{\pi}{4}\right)\right)\right]$$

$$= \frac{8}{3}\left[\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)\right]$$

$$= \frac{16}{3}\sqrt{2}.$$

Lecture 29: More general regions in polar coordinates

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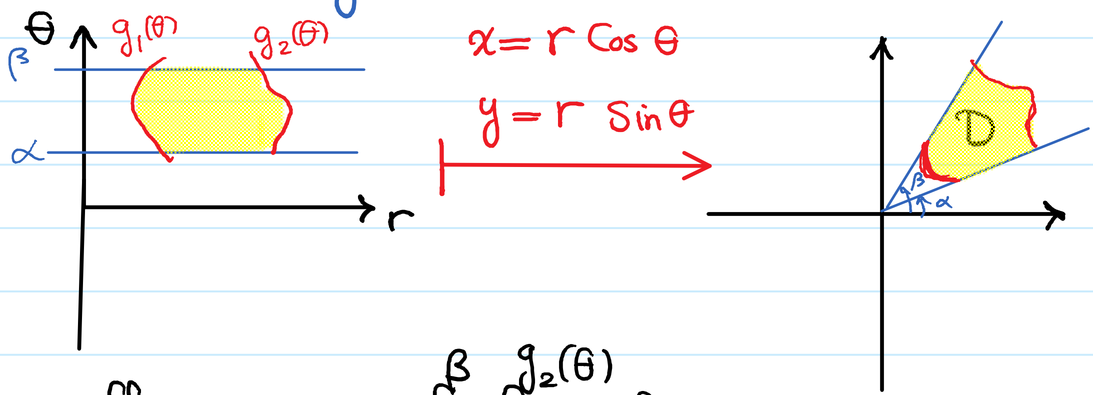
So far we have worked only with rectangles in polar coordinates.

Another type of regions that are extremely useful are the "r-simple" regions: $\alpha \le \theta \le \beta$, $g_1(\theta) \le r \le g_2(\theta)$.

(that's why when you use polar coordinates, you should always start with the range for θ.)



$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$