Math 21C Midterm II, Fall 02, Lindblad.

1. A particle moves with position vector given by $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t^{3/2}\mathbf{k}$.
(a) Find the velocity and speed of the particle at time $t = \pi$.
(b) How far does the particle travel between time $t = 0$ and $t = 10$?

2. Let $F(x, y) = x^3 - x^2 + y^2 - y + 1$
(a) Find the gradient $\nabla F(x, y)$.
(b) Find the directional derivative in the direction of $\langle 1, 2\rangle$ at the point $(2, 3)$.
(c) Let $\mathbf{r}(t) = \langle f(t), g(t)\rangle$ be a curve such that $\mathbf{r}(0) = \langle 1, 0\rangle$ and $\mathbf{r}'(0) = \langle 1, 1\rangle$. Let $h(t) = F(\mathbf{r}(t))$. Find $h'(0)$.

3. Consider the surface given by $z = f(x, y) = \sqrt{x^2 + 3y^2}$.
a) Find the tangent plane to the surface at the point $(1, 1, 2)$.
b) A student was asked to find an approximation for $f(1.1, 1.2)$ but the professor did not allow calculators. The student noticed that $f(1.1, 1.2)$ is approximately $f(1, 1) = \sqrt{1 + 3} = 2$. Use the linear approximation to get a better approximation.

4. Let $f(x, y) = x^3 - x^2 + y^2 - y + 1$. Find the critical points of $f(x, y)$ and determine if they are local max, min or saddle points. Are there any absolute max or min?

5. Use Lagrange multipliers to find the point on the hyperboloid $\{(x, y, z);\ z^2 = x^2 + y^2 + 1,\ z \geq 0\}$ that is closest to the point $(0, 0, -2)$. (NOT PART OF OUR EXAM)

1.(a) $\vec{r}(t) = \langle \cos t, \sin t, t^{3/2} \rangle$

$$\vec{v}(t) = \vec{r}\,'(t) = \langle -\sin t, \cos t, \tfrac{3}{2} t^{1/2} \rangle$$

$$s(t) = \|\vec{v}(t)\| = \sqrt{(-\sin t)^2 + \cos^2 t + \left(\tfrac{3}{2} t^{1/2}\right)^2}$$

$$= \sqrt{1 + \tfrac{9}{4} t}$$

$$\Rightarrow \vec{v}(\pi) = \langle 0, -1, \tfrac{3}{2}\sqrt{\pi} \rangle$$

$$s(\pi) = \sqrt{1 + \frac{9\pi}{4}}.$$

(b) Total length $= \int_a^b \|\vec{r}\,'(t)\|\, dt$

$$= \int_0^{10} \sqrt{1 + \tfrac{9}{4} t}\; dt$$

$u = 1 + \frac{9}{4} t$

$du = \frac{9}{4} dt$

$$= \int_1^{\frac{45}{2}} (\sqrt{u})\left(\tfrac{4}{9}\right) du$$

$$= \frac{4}{9} \cdot \frac{2}{3} \cdot u^{3/2} \Big|_1^{45/2}$$

$$= \frac{8}{27}\left(\frac{45}{2}\sqrt{\frac{45}{2}} - 1\right)$$

$$= \frac{20}{3}\sqrt{\frac{45}{2}} - \frac{8}{27}.$$

2. (a) $F(x,y) = x^3 - x^2 + y^2 - y + 1$

$\nabla F = \langle 3x^2 - 2x \, , \, 2y - 1 \rangle$

(b) $D_u F = \vec{u} \cdot \nabla F$ where $\vec{u} = \dfrac{\vec{v}}{\|\vec{v}\|}$.

$\Rightarrow \vec{u} = \dfrac{\langle 1, 2 \rangle}{\sqrt{1+4}} = \left\langle \dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}} \right\rangle$.

$\nabla F(2,3) = \langle 8, 5 \rangle$

$\Rightarrow D_u F(2,3) = \left\langle \dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}} \right\rangle \cdot \langle 8, 5 \rangle$

$= \dfrac{18}{\sqrt{5}} = \dfrac{18\sqrt{5}}{5}$.

(c) $\vec{r}(0) = \langle 1, 0 \rangle$ and $\vec{r}\,'(0) = \langle 1, 1 \rangle$.

$\dfrac{d}{dt} f(\vec{r}(t)) \Big|_{t=0} = \vec{r}\,'(0) \cdot \nabla f(\vec{r}(0))$

$= \langle 1, 1 \rangle \cdot \nabla f(1, 0)$

$= \langle 1, 1 \rangle \cdot \langle 1, -1 \rangle = 0$.

3. $z = f(x,y) = \sqrt{x^2+3y^2}$.

(a) Tangent plane at $(1,1,2)$

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

$$\nabla f = \frac{1}{2\sqrt{x^2+3y^2}} \langle 2x, 6y \rangle = \frac{1}{\sqrt{x^2+3y^2}} \langle x, 3y \rangle$$

$$\Rightarrow \nabla f(1,1) = \frac{1}{2} \langle 1, 3 \rangle.$$

$$\Rightarrow z = 2 + \frac{1}{2}(x-1) + \frac{3}{2}(y-1)$$

$$\Rightarrow z = \frac{x}{2} + \frac{3}{2} y$$

(b) $f(1.1, 1.2) \approx ?$

$$f(x_0+\Delta x, y_0+\Delta y) \approx f(x_0,y_0) + f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y$$

$$f(1.1, 1.2) \approx 2 + \frac{1}{2}(0.1) + \frac{3}{2}(0.2)$$

$$= 2 + 0.05 + 0.3$$

$$= 2.35$$

4. $f(x,y) = x^3 - x^2 + y^2 - y + 1$

$\nabla f = \langle 3x^2 - 2x, 2y - 1 \rangle = \langle 0, 0 \rangle$

$\Rightarrow \begin{cases} 3x^2 - 2x = 0 \\ 2y - 1 = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \text{ or } x = \frac{2}{3} \\ y = \frac{1}{2}. \end{cases}$

$\Rightarrow$ critical pts are $(0, \frac{1}{2})$ and $(\frac{2}{3}, \frac{1}{2})$.

$\left.\begin{array}{l} f_{xx} = 6x - 2 \\ f_{xy} = 0 \\ f_{yy} = 2 \end{array}\right\} \Rightarrow D = f_{xx} f_{yy} - f_{xy}^2$

$\Rightarrow D = (6x-2)(2) = 4(3x-1)$.

Critical Pts	D	f_{xx}	Result
$(0, 1/2)$	$-$		$\rightarrow$ Saddle pt
$(2/3, 1/2)$	$+$	$+$	local min

- No, there is neither an absolute max nor an absolute min. Since $f(x,0) = x^3 - x^2 + 1$ is an degree polynomial, the range of f is $(-\infty, +\infty)$. So there is no abs. max or abs. min.

5. $z^2 = x^2+y^2+1$, $z \geq 0$

closest to $(0,0,-2)$

$\Rightarrow$ min of $f(x,y,z) = x^2+y^2+(z+2)^2$

under the condition :

$$g(x,y,z) = x^2+y^2-z^2 = -1$$

and $z \geq 0$.

We have to solve the following system of equations:

$$\begin{cases} \nabla f = c\,\nabla g \\ g = -1 \\ z \geq 0 \end{cases}$$

$$\Rightarrow \langle 2x, 2y, 2(z+1)\rangle = c\,\langle 2x, 2y, -2z\rangle$$

$$\Rightarrow \begin{cases} 2x = 2cx & \Rightarrow x=0 \ \underline{\text{or}}\ c=1 \\ 2y = 2cy & \Rightarrow y=0 \ \underline{\text{or}}\ c=1 \\ 2z+2 = -2cz & \end{cases}$$

<u>If $c=1$, then</u> $2z+2 = -2z \Rightarrow z = -\frac{1}{2}$

NOT possible as $z \geq 0$.

<u>So $c \neq 1$</u> . Hence $\left.\begin{array}{l} x=y=0 \\ x^2+y^2-z^2=-1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} z^2=1 \\ z\geq 0 \end{array}\right\} \Rightarrow z=1$

$\Rightarrow (0,0,1)$ is the closest pt to $(0,0,-2)$.