Name: ______________________________
(Use capitals)
Student number: ______________________________

Math 20C
Final Exam
July 31, 2004

Read each question carefully, and answer each question completely.
Show all of your work. No credit will be given for unsupported answers.
Write your solutions clearly and legibly. No credit will be given for illegible solutions.

1. (8 points)

(a) Find the angle between the planes $2x + y + 3z = 1$ and $-x - 3y + 2z = 5$.

$n_1 = \langle 2, 1, 3 \rangle$ and $n_2 = \langle -1, -3, 2 \rangle$: normal vectors

$$\left.\begin{array}{l} n_1 \cdot n_2 = -2 - 3 + 6 = 1 \\ \|n_1\| = \|n_2\| = \sqrt{1+4+9} = \sqrt{14} \end{array}\right\} \Rightarrow \cos\theta = \frac{n_1 \cdot n_2}{\|n_1\| \, \|n_2\|} = \frac{1}{14}$$

$\Rightarrow \theta = \operatorname{Arccos}(1/14)$.

(b) Find a vector parallel to the line of intersection of the planes given in (a).

$\vec{n_1} \times \vec{n_2} \perp n_1$ and $n_2 \Rightarrow n_1 \times n_2$ is parallel to the line of intersection.

$$n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 2 & 1 & 3 \\ -1 & -3 & 2 \end{vmatrix} = \left\langle \begin{vmatrix} 1 & 3 \\ -3 & 2 \end{vmatrix}, -\begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ -1 & -3 \end{vmatrix} \right\rangle$$

$$= \langle 11, -7, -5 \rangle$$

#	Score
1	
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2. (8 points)

A particle has the velocity function $\vec{v}(t) = \langle -3\sin(t), 4, 3\cos(t)\rangle$.

(a) Find the particle acceleration $\vec{a}(t)$.

$$\vec{a}(t) = \vec{v}\,'(t) = \langle -3\cos(t), 0, -3\sin(t)\rangle.$$

(b) The particle initial position is given by $\vec{r}(0) = \langle 3, 1, 0\rangle$. Find the particle position function $\vec{r}(t)$.

$$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(u)\,du$$

$$= \langle 3,1,0\rangle + \int_0^t \langle -3\sin u, 4, 3\cos u\rangle\,du$$

$$= \langle 3,1,0\rangle + \langle 3\cos u, 4u, 3\sin u\rangle\Big|_0^t$$

$$= \langle 3,1,0\rangle + \langle 3\cos t - 3, 4t, 3\sin t\rangle$$

$$= \langle 3\cos t, 4t+1, 3\sin t\rangle$$

(c) Reparametrize the curve $\vec{r}(t)$ with respect to the arc length measured from the point where $t = 0$, in the direction of increasing t.

NOT part of our exam.

Total distance travelled till t

$$= \int_0^t \|\vec{v}(u)\|\,du = \int_0^t \sqrt{9+16}\,du = 5t$$

$\vec{r}_{arc}(s) = \vec{r}(t_s)$ where t_s is the time when the total travelled distance is $s \Rightarrow s = 5t_s \Rightarrow t_s = s/5$

$$\vec{r}_{arc}(s) = \vec{r}(s/5) = \langle 3\cos(s/5), \tfrac{4}{5}s+1, 3\sin(s/5)\rangle.$$

3. (8 points)

(a) Find an equation for the plane tangent to the graph of $f(x,y) = \frac{1}{\pi}\cos\left(\frac{\pi}{2}x^2y\right)$ at the point $(-1,1)$.

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$f_x = \left(\frac{1}{\pi}\right)\left(\frac{\pi}{2}y\right)(2x)\left(-\sin\left(\frac{\pi}{2}x^2y\right)\right) = -xy\sin\left(\frac{\pi}{2}x^2y\right)$$

$$f_y = \left(\frac{1}{\pi}\right)\left(\frac{\pi}{2}x^2\right)\left(-\sin\left(\frac{\pi}{2}x^2y\right)\right) = -\frac{1}{2}x^2\sin\left(\frac{\pi}{2}x^2y\right)$$

$$z = (x+1) - \frac{1}{2}(y-1)$$

$$\Rightarrow z = x - \frac{y}{2} + \frac{3}{2}$$

(b) Find the linear approximation for $f(-1.1, 1.2)$.

$$f(-1.1, 1.2) \approx (-1.1) - \frac{1.2}{2} + \frac{3}{2}$$

$$= -1.1 - 0.6 + 1.5$$

$$= -0.2$$

4. (8 points)

Consider the function $f(x, y, z) = 2\sin(xyz)$.

(a) Find a unit vector in the direction of greatest increase of f at the point $(\pi, 1, 1)$.

Direction of the greatest increase is the direction of ∇f.

$$\nabla f = \langle 2yz \cos(xyz), 2xz \cos(xyz), 2xy \cos(xyz) \rangle$$

$$\nabla f(\pi, 1, 1) = -2 \langle 1, \pi, \pi \rangle = \langle -2, -2\pi, -2\pi \rangle$$

$$\text{unit vector} = \frac{-1}{\sqrt{1+2\pi^2}} \langle 1, \pi, \pi \rangle$$

$$= \left\langle \frac{-1}{\sqrt{2\pi^2+1}}, \frac{-\pi}{\sqrt{2\pi^2+1}}, \frac{-\pi}{\sqrt{2\pi^2+1}} \right\rangle.$$

(b) Find the directional derivative of f at the point $(\pi, 1, 1)$ in the direction given by the vector $\langle 1, 1, -1 \rangle$.

$$D_u f(P_0) = \nabla f(P_0) \cdot \vec{u} = \frac{\nabla f(P_0) \cdot \vec{v}}{\|\vec{v}\|}$$

$$= \frac{\langle -2, -2\pi, -2\pi \rangle \cdot \langle 1, 1, -1 \rangle}{\sqrt{3}}$$

$$= \frac{-2}{\sqrt{3}} = \frac{-2\sqrt{3}}{3}.$$

This problem is flawed. Nevertheless it is a good exercise.

5. (8 points)

Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = xy$ subject to the constraint $2x + y^2 = 3$.

$$\begin{cases} \nabla f = c\nabla g \\ g = 3 \end{cases} \Rightarrow \begin{cases} \langle y, x\rangle = c\langle 2, 2y\rangle \\ 2x + y^2 = 3 \end{cases}$$

$$\Rightarrow \begin{cases} y = 2c \\ x = 2cy \\ 2x + y^2 = 3 \end{cases} \Rightarrow \begin{cases} x = y^2 \\ 2x + y^2 = 3 \end{cases} \Rightarrow 3x = 3 \Rightarrow x = 1.$$

$$\Rightarrow y = \pm 1$$

So if f has a max and a min, they should occur at either $(1,1)$ or $(1,-1)$.

$f(1,1) = 1$ and $f(1,-1) = -1 \Rightarrow$ if min and max exist, $f(1,1) = 1$ is max and $f(1,-1) = -1$ is the min.

Notice $2x + y^2 = 3 \Rightarrow x = \frac{3}{2} - \frac{y^2}{2}$

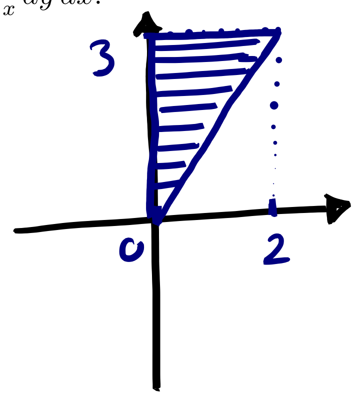
$$\Rightarrow f(x,y) = \left(\frac{3}{2} - \frac{y^2}{2}\right) y = -\frac{y^3}{2} + \frac{3}{2}y$$

$$\Rightarrow \lim_{y \to +\infty} -\frac{y^3}{2} + \frac{3}{2}y = -\infty$$ there is no min

$$\lim_{y \to -\infty} -\frac{y^3}{2} + \frac{3}{2}y = +\infty$$ there is no max

6. (8 points)

(a) Sketch the region of integration, D, whose area is given by the double integral $\int\int_D dA = \int_0^2 \int_{\frac{3}{2}x}^3 dy\, dx$.



(b) Compute the double integral given in (a).

$$A(x) = \int_{3/2\,x}^{3} dy = y\Big|_{3/2\,x}^{3} = 3 - \frac{3}{2}x$$

$$\Rightarrow \iint_D dA = \int_0^2 3 - \frac{3}{2}x \; dx = \left(3x - \frac{3}{4}x^2\right)\Big|_0^2$$

$$= 6 - 3 = 3.$$

[Second solution: $\iint_D dA = \text{area}(D) = \frac{1}{2}(2)(3) = 3.$]

(c) Change the order of integration in the integral given in (a). (You don't need to compute the integral again.)

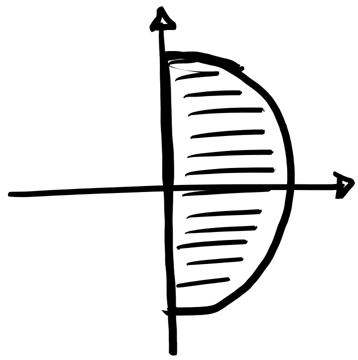
$$\int_0^3 \int_0^{\frac{2}{3}y} dx\, dy$$

$$\boxed{y = \frac{3}{2}x \Rightarrow x = \frac{2}{3}y}$$

7. (8 points)

Use polar coordinates to compute the double integral of $f(x, y) = xy$ in the region $D = \{(x, y) : \quad 0 \leq x, \quad x^2 + y^2 \leq 4\}$.

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and



$0 \leq r \leq 2$

$$\iint_D xy \, dA = \int_{-\pi/2}^{\pi/2} \int_0^2 (r\cos\theta)(r\sin\theta)\, r \, dr \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^2 r^3 \cos\theta \sin\theta \, dr \, d\theta$$

$$A(\theta) = \int_0^2 r^3 \cos\theta \sin\theta \, dr$$

$$= \left(\frac{1}{4} r^4 \cos\theta \sin\theta\right)\Big|_0^2$$

$$= 4 \cos\theta \sin\theta.$$

$$\Rightarrow \iint_D xy \, dA = \int_{-\pi/2}^{\pi/2} 4 \cos\theta \sin\theta \, d\theta$$

$u = \sin\theta$
$\Rightarrow du = \cos\theta \, d\theta$

$$= \int_{-1}^{1} 4u \, du = 2u^2 \Big|_{-1}^{1} = 0$$

8. (8 points)

~~Use a triple integral to compute the volume of the tetrahedron whose sides are given by the planes $x = 0$, $y = 0$, $z = 0$, and $2x + 2y + z = 2$.~~

NOT PART OF OUR EXAM.