Math 21C Final, Fall 02, Lindblad.

1. A particle moves with position vector given by $\mathbf{r}(t) = t^2\,\mathbf{i} + (1-t^2)\,\mathbf{j} + t^3\mathbf{k}$.
(a) Find the equation of the tangent line to the curve at $t = 1$!
(b) What distance does the particle travel between time $t = 0$ and $t = 1$?

2. Given the three points $P(0,1,1)$, $Q(0,2,0)$ and $R(2,1,0)$.
(a) Find the equation of the plane containing the three points!
(b) Find the Area of the triangle with the three points as corners!
(c) What are the cosines of the angles at the three corners of the triangle in (b)?

3. The temperature of a metal ball centered at the origin is given by $T = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$.
(a) Find the rate of change of T at $(1,2,2)$ in the direction toward the point $(2,1,3)$
(b) Explain why for any point (x,y,z) in the ball, the direction of greatest increase in the temperature is toward the center of the ball.

4. (a) Find the points on the ellipsoid $4x^2 + y^2 + z^2 = 4$ where the tangent plane is parallel to the plane $x + 2y - z = 0$.
(b) Find the equation for the tangent plane at these points.

5. (a) Find the critical points of $f(x,y) = x^2 + 5y^2 + 3xy + 8$ and determine if they are local max, min or saddle points.
(b) Find the max and min of $f(x,y)$ on the set $g(x,y) = x^2 + y^2 = 4$.
(c) Find the absolute max and min of $f(x,y)$ over the set $D = \{(x,y)|g(x,y) \le 4\}$.

6. UPS charges according to weight as well as size, for sending a box. If $x, y, z \ge 0$ are the lengths of the edges of a rectangular box, then in order that the package should not be over sized we must have $G(x,y,z) = 2x + 2y + z \le 120$ inches. Find the maximum of the volume $V(x,y,z) = xyz$ for a package that is not over sized.

7. (a) Write the iterated integral $\displaystyle\int_0^1 \int_1^{1/y} x^2 e^{-x^2}\,dx\,dy$ as a double integral over some unbounded domain D in the x-y plane.
(b) Evaluate the integral in (a) by changing the order of integration.

8. Find $\displaystyle\iint_R (1+y)\cos(x^2+y^2)\,dA$, where $R = \{(x,y);\, 1 \le x^2+y^2 \le 4\}$.

9. Find the area of the part of the plane $x + 2y + z = 8$ above the region $D = \{(x,y);\, 0 \le x \le 1,\, 2x^2 \le y \le 1 + x^2\}$, i.e. of the surface $S = \{(x,y,z);\, (x,y) \in D,\, x + 2y + z = 8\}$.

NOT PART OF OUR EXAM

10. Let D be the triangle in the x-y plane with vertices $(0,0)$, $(0,4)$ and $(1,2)$. Write the volume of the solid E, below the plane $z = x - y + 20$ and above the triangle D, as a double integral over D and evaluate it.
(i.e. $E = \{(x,y,z)|\ 0 \le z \le x - y + 20,\ (x,y) \in D\}$.)

1. (a) Equation of the tangent line: $\vec{r}'(t_0)\,t + \vec{r}(t_0)$

$r'(t) = \langle 2t, -2t, 3t^2 \rangle \Rightarrow r'(1) = \langle 2, -2, 3 \rangle$

$r(1) = \langle 1, 0, 1 \rangle$.

So the vector equation of the tangent line is

$$t\,\langle 2,-2,3\rangle + \langle 1,0,1\rangle = \langle 2t+1, -2t, 3t+1\rangle$$

(b) <u>Speed</u>: $s(t) = \|r'(t)\| = \sqrt{(2t)^2 + (-2t)^2 + (3t^2)^2}$

$$= \sqrt{8t^2 + 9t^4} = |t|\sqrt{8+9t^2}$$

<u>Total distance</u>
<u>$0 \le t \le 1$</u>: $\displaystyle\int_0^1 s(t)\,dt = \int_0^1 t\sqrt{8+9t^2}\,dt$

$$u = 8 + 9t^2 \Rightarrow du = 18\,t\,dt$$

$$\Rightarrow t\sqrt{8+9t^2}\,dt = \frac{1}{18}\sqrt{u}\,du$$

Total distance
$0 \le t \le 1$ $\displaystyle = \int_8^{17} \frac{1}{18}\sqrt{u}\,du$

$$= \frac{1}{18}\cdot\frac{2}{3}\cdot u^{3/2}\Big|_8^{17}$$

$$= \frac{1}{27}\left(17\sqrt{17} - 16\sqrt{2}\right).$$

2. $P = (0,1,1)$, $Q = (0,2,0)$, $R = (2,1,0)$

(a) normal vector $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$

$$= \langle 0, 1, -1 \rangle \times \langle 2, 0, -1 \rangle$$

$$= \begin{vmatrix} i & j & k \\ 0 & 1 & -1 \\ 2 & 0 & -1 \end{vmatrix}$$

$$= \left\langle \begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix}, -\begin{vmatrix} 0 & -1 \\ 2 & -1 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} \right\rangle$$

$$= \langle -1, -2, -2 \rangle$$

$$\Rightarrow -x - 2y - 2z = (-1)(0) + (-2)(1) + (-2)(1)$$

$$= -4$$

$$\Rightarrow x + 2y + 2z = 4 .$$

(b) Area of $PQR = \frac{1}{2} \| \overrightarrow{PQ} \times \overrightarrow{PR} \|$

$$= \frac{1}{2} \sqrt{1 + 4 + 4} = \frac{3}{2} .$$

(b) $\| \overrightarrow{PQ} \| = \| \langle 0, 1, -1 \rangle \| = \sqrt{2}$

$\| \overrightarrow{QR} \| = \| \langle 2, -1, 0 \rangle \| = \sqrt{5}$

$\| \overrightarrow{RP} \| = \| \langle -2, 0, 1 \rangle \| = \sqrt{5}$

$$\cos(Q\hat{P}R) = \frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{\|\overrightarrow{PQ}\|\,\|\overrightarrow{PR}\|} = \frac{1}{\sqrt{2}\cdot\sqrt{5}} = \frac{\sqrt{10}}{10}.$$

$$\cos(P\hat{R}Q) = \frac{\overrightarrow{RP}\cdot\overrightarrow{RQ}}{\|\overrightarrow{RP}\|\,\|\overrightarrow{RQ}\|} = \frac{4}{\sqrt{5}\cdot\sqrt{5}} = \frac{4}{5}$$

Either use the fact that it is an isosceles, or

$$\cos(R\hat{Q}P) = \frac{\overrightarrow{QR}\cdot\overrightarrow{QP}}{\|\overrightarrow{QR}\|\,\|\overrightarrow{QP}\|} = \frac{1}{\sqrt{5}\cdot\sqrt{2}} = \frac{\sqrt{10}}{10}.$$

3.(a) Rate of change in the direction of $\vec{v}$ $= D_{\vec{u}} f = \dfrac{\nabla f(P_0)\cdot\vec{v}}{\|\vec{v}\|}$

$$\nabla f = \left\langle \frac{-x}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{-z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$

$$\Rightarrow \nabla f(1,2,2) = \left\langle \frac{-1}{27}, \frac{-2}{27}, \frac{-2}{27} \right\rangle$$

The direction towards the point $(2,1,3)$ is the same of the direction of the vector $\langle 2,1,3\rangle - \langle 1,2,2\rangle = \langle 1,-1,1\rangle$. So

Rate of change in the direction of $\langle 1,-1,1\rangle$ $= \dfrac{\nabla f(1,2,2)\cdot\langle 1,-1,1\rangle}{\|\langle 1,-1,1\rangle\|} = \dfrac{-1}{27\sqrt{3}}(1-2+2) = \dfrac{-1}{27\sqrt{3}}$

(b) The direction of the greatest increase is the direction of $\nabla f(p)$. As we have seen

$$\nabla f(P) = \frac{\overrightarrow{PO}}{\|\overrightarrow{PO}\|^3}$$

which is in the direction of the origin.

4.(a) $\nabla f(P_0)$ is a normal vector of the tangent plane of the level surface $f(x,y,z) = f(P_0)$ at P_0.

And two planes are parallel if and only if their normal vectors are parallel.

$\Rightarrow$ Let $f(x,y,z) = 4x^2+y^2+z^2$. Then we should find (x,y,z) s.t.

$$\nabla f(x,y,z) = \langle 8x, 2y, 2z \rangle$$

is parallel to a normal vector of $x+2y-z=0$.

$\Rightarrow$ for some c, $\langle 8x, 2y, 2z\rangle = c\langle 1, 2, -1\rangle$

$$\Rightarrow \begin{cases} 8x = c \\ 2y = 2c \\ 2z = -c \\ 4x^2+y^2+z^2 = 4 \end{cases} \Rightarrow \begin{cases} y = 8x \\ z = -4x \\ 4x^2+y^2+z^2 = 4 \end{cases}$$

$$\Rightarrow 4x^2 + (8x)^2 + (-4x)^2 = 4$$

$$\Rightarrow (4+64+16)\,x^2 = 4$$

$$\Rightarrow x^2 = \frac{1}{21} \Rightarrow x = \pm\sqrt{\frac{1}{21}}$$

$\Rightarrow$ points are $x(1, 8, -4)$

$$= \pm\frac{1}{\sqrt{21}}(1,8,-4).$$

(b) Eq. of the tangent plane at (x_0, y_0, z_0) is

$$\nabla f(x_0, y_0, z_0) \cdot \langle x, y, z\rangle = \nabla f(x_0, y_0, z_0) \cdot \langle x_0, y_0, z_0\rangle$$

$$\Rightarrow 8x_0 x + 2y_0 y + 2z_0 z = 8x_0^2 + 2y_0^2 + 2z_0^2$$

$$\Rightarrow 4x_0 x + y_0 y + z_0 z = 4x_0^2 + y_0^2 + z_0^2 = 4$$

$\Rightarrow$ At $\frac{1}{\sqrt{21}}(1,8,-4)$:

$$4x + 8y - 4z = 4\sqrt{21}\ , \quad \text{and}$$

at $\frac{-1}{\sqrt{21}}(1,8,-4)$:

$$4x + 8y - 4z = -4\sqrt{21}.$$

5. (a) $\nabla f = \langle 2x+3y, 10y+3x \rangle = \langle 0,0 \rangle$

$$\Rightarrow \begin{cases} 2x+3y=0 \\ 3x+10y=0 \end{cases} \begin{aligned} &\Rightarrow y = -\frac{2}{3}x \\ &\Rightarrow \left(3-\frac{20}{3}\right)x = 0 \\ &\Rightarrow x=0 \Rightarrow y=0 \end{aligned}$$

$\Rightarrow$ there is only one critical pt $(0,0)$.

$f_{xx} = 2$, $f_{xy} = 3$, $f_{yy} = 10 \Rightarrow D = f_{xx} \cdot f_{yy} - (f_{xy})^2$

$= 20 - 9 = 11 > 0$

$D > 0$, $f_{xx} > 0 \Rightarrow (0,0)$ is a local min.

(b) $x^2+y^2=4$ is closed and bounded, and f is continuous $\Rightarrow f$ has a max and a min subject to the constraint $x^2+y^2=4$. So Lagrange multipliers method give us the answer.

First Solution

$$\begin{cases} \nabla f = c\nabla g \\ g = 4 \end{cases} \Rightarrow \begin{cases} \langle 2x+3y, 10y+3x \rangle = c\langle 2x, 2y \rangle \\ x^2+y^2=4 \end{cases}$$

(where $g(x,y) = x^2+y^2$) $\Rightarrow \begin{cases} 2x+3y = 2cx \\ 3x+10y = 2cy \\ x^2+y^2 = 4 \end{cases}$

- If $x=0 \Rightarrow 3y=0 \Rightarrow y=0$ which is NOT possible.
- If $y=0 \Rightarrow 3x=0$ " " " " .

So if $x\neq 0$ and $y\neq 0$, then

$$c = \frac{2x+3y}{2x} = \frac{3x+10y}{2y}$$

$$\Rightarrow 2xy+3y^2 = 3x^2+10xy \Rightarrow x^2+\frac{8}{3}xy-y^2=0$$

$$\Rightarrow \left(\frac{x}{y}\right)^2+\frac{8}{3}\left(\frac{x}{y}\right)-1=0 \Rightarrow \left(\frac{x}{y}\right)^2+\frac{8}{3}\left(\frac{x}{y}\right)+\frac{16}{9}=1+\frac{16}{9}$$

$$\Rightarrow \left[\left(\frac{x}{y}\right)+\frac{4}{3}\right]^2=\frac{25}{9} \Rightarrow \frac{x}{y}=-\frac{4}{3}\pm\frac{5}{3}$$

$$\Rightarrow \frac{x}{y}=-3 \text{ or } \frac{1}{3} \Rightarrow x=-3y \text{ or } 3x=y$$

$$\left.\begin{array}{l} x=-3y \\ x^2+y^2=4 \end{array}\right\} \Rightarrow 9y^2+y^2=4 \Rightarrow y=\pm\sqrt{\frac{4}{10}}=\pm\frac{\sqrt{10}}{5}$$

$$\Rightarrow \left(\frac{-3\sqrt{10}}{5}, \frac{\sqrt{10}}{5}\right) \text{ or } \left(\frac{3\sqrt{10}}{5}, \frac{-\sqrt{10}}{5}\right)$$

$$\left.\begin{array}{l} 3x=y \\ x^2+y^2=4 \end{array}\right\} \Rightarrow x^2+9x^2=4 \Rightarrow x=\pm\sqrt{\frac{4}{10}}=\pm\frac{\sqrt{10}}{5}$$

$$\Rightarrow \left(\frac{\sqrt{10}}{5}, \frac{3\sqrt{10}}{5}\right) \text{ or } \left(\frac{-\sqrt{10}}{5}, \frac{-3\sqrt{10}}{5}\right)$$

Relevant Pts	$\pm\left(\frac{3\sqrt{10}}{5}, \frac{-\sqrt{10}}{5}\right)$	$\pm\left(\frac{\sqrt{10}}{5}, \frac{3\sqrt{10}}{5}\right)$
Value of f	$\underbrace{10}_{\text{min}}$	$\underbrace{30}_{\text{max}}$

Second Solution

$$\begin{cases} (2-2c)x + 3y = 0 & \Rightarrow y = \frac{-2+2c}{3}x \\ 3x + (10-2c)y = 0 \end{cases}$$

$$\Rightarrow -3x + (10-2c)\frac{(2-2c)}{3}x = 0$$

$$\Rightarrow (11 - 24c + 4c^2)x^2 = 0$$

$\Rightarrow$ either $x = 0$ or $4c^2 - 24c + 11 = 0$

If $x = 0$, then $y = 0$ which is not possible as $x^2 + y^2 = 4$.

$$\Rightarrow (2c - 11)(2c - 1) = 0 \Rightarrow c = \frac{1}{2} \text{ or } c = \frac{11}{2}$$

$$\Rightarrow y = \frac{-1}{3}x \quad \text{or} \quad y = 3x.$$

$$\Rightarrow x^2 + y^2 = 9y^2 + y^2 = 4 \Rightarrow y^2 = \frac{2}{5} \Rightarrow \frac{\pm 2}{\sqrt{10}}(-3, 1)$$

Or $\quad x^2 + y^2 = x^2 + 9x^2 = 4 \Rightarrow x^2 = \frac{2}{5} \Rightarrow \pm\frac{2}{\sqrt{10}}(1, 3)$

$\Rightarrow$

(x, y)	$\pm\frac{2}{\sqrt{10}}(-3, 1)$	$\pm\frac{2}{\sqrt{10}}(1, 3)$
$x^2+5y^2+3xy+8$	$\underbrace{10}_{\text{min}}$	$\underbrace{30}_{\text{max}}$

(c) Global max and min of a contin. function on a bounded and closed set occurs either at a critical pt in the interior or at the boundary

Relevant pts	$\pm\frac{2}{\sqrt{10}}(-3, 1)$	$\pm\frac{2}{\sqrt{10}}(1, 3)$	$(0, 0)$
$f(x, y)$	10	$\underbrace{30}_{\text{max}}$	$\underbrace{8}_{\text{min}}$

6. Find max of xyz in the region

$0 \le x, y, z$ and $2x + 2y + z \le 120$.

• Global max and min of a conti. function on a bounded and closed set occurs either at a critical pt in the interior, or at the boundary:

$\nabla f = \langle yz, xz, xy \rangle = \langle 0, 0, 0 \rangle$

$\Rightarrow \begin{cases} yz = 0 \\ xz = 0 \\ xy = 0 \end{cases} \Rightarrow$ at least two of the components are zero $\Rightarrow$ there is NO critical point in the interior of the given region.

On $x=0 \Rightarrow xyz=0$

On $y=0 \Rightarrow xyz=0$

On $z=0 \Rightarrow xyz=0$

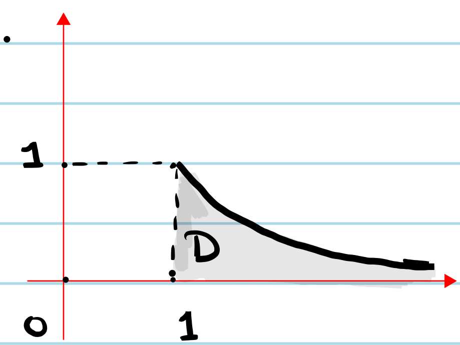
So we have to find max of xyz subject to the constraint $2x+2y+z=120$, $x,y,z>0$

$$\Rightarrow \begin{cases} \langle yz, xz, xy\rangle = c\langle 2,2,1\rangle \\ 2x+2y+z=120 \end{cases}$$

$$\Rightarrow \begin{cases} yz=2xy & \xrightarrow{y>0} & z=2x \\ xz=2xy & \xrightarrow{x>0} & z=2y \\ 2x+2y+z=120 \end{cases}$$

$$\Rightarrow \quad 2x+2x+2x=120 \Rightarrow x=20$$

$$\Rightarrow (x,y,z)=(20,20,40).$$



7. $\displaystyle\int_0^1\int_1^{1/y} x^2 e^{-x^2}\,dx\,dy$

$$=\iint_D x^2 e^{-x^2}\,dA$$

$$= \int_1^{\infty}\int_0^{1/x} x^2 e^{-x^2}\, dy\, dx$$

$$\boxed{x = 1/y \Rightarrow y = 1/x}$$

$$A(x) = \int_0^{1/x} x^2 e^{-x^2}\, dy = x^2 e^{-x^2} y \Big|_0^{1/x}$$

$$= x e^{-x^2}$$

$$\Rightarrow \iint_D x^2 e^{-x^2}\, dA = \int_1^{\infty} x e^{-x^2}\, dx = \lim_{R\to\infty} \int_1^{R} x e^{-x^2}\, dx$$

$$\boxed{\begin{aligned} u = x^2 &\Rightarrow du = 2x\, dx \\ &\Rightarrow x e^{-x^2} dx = \frac{e^{-u}}{2}\, du \end{aligned}}$$

$$= \lim_{R\to\infty} \int_1^{R^2} \frac{e^{-u}}{2}\, du$$

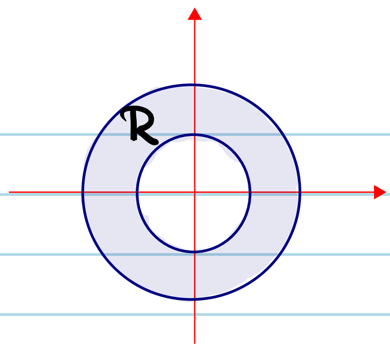
$$= \lim_{R\to\infty} \frac{-e^{-u}}{2} \Big|_1^{R^2}$$

$$= \lim_{R\to\infty} \frac{\frac{1}{e} - e^{-R^2}}{2}$$

$$= \frac{1}{2e}.$$

$$\Rightarrow \iint_D x^2 e^{-x^2}\, dA = \frac{1}{2e}.$$

$$8.\ \iint_R (1+y)\cos(x^2+y^2)\, dA$$



$0 \le \theta \le 2\pi, \quad 1 \le r \le 2$

$$\Rightarrow \iint_R (1+y)\cos(x^2+y^2)\, dA$$

$$= \int_0^{2\pi}\int_1^2 (1+r\sin\theta)\cos(r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_1^2 r\cos(r^2)\, dr\, d\theta + \int_0^{2\pi}\int_1^2 \sin(\theta)\, r^2\cos(r^2)\, dr\, d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_1^2 r\cos(r^2)\, dr\right) + \left(\int_0^{2\pi}\sin\theta\, d\theta\right)\left(\int_1^2 r^2\cos(r^2)\, dr\right)$$

$$\boxed{u = r^2 \Rightarrow du = 2r\,dr}$$

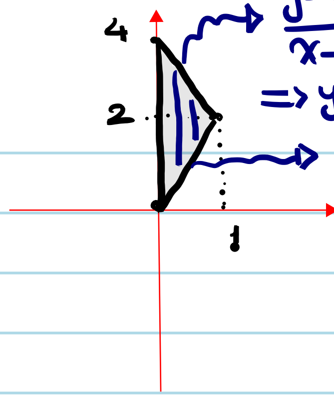
$\Downarrow$

$$= 2\pi \int_1^4 \frac{1}{2}\cos(u)\, du + [-\cos\theta]\Big|_0^{2\pi}\left(\int_1^2 r^2\cos(r^2)\, dr\right)$$

$$= \pi \sin(u)\Big|_1^4 + (0)\left(\int_1^2 r^2\cos(r^2)\, dr\right)$$

$$= \pi(\sin(4) - \sin(1)).$$

10. vol. $= \iint_D x - y + 20 \, dA$



$\dfrac{y-4}{x-0} = \dfrac{2-4}{1-0} = -2$

$\Rightarrow y = -2x + 4$

$y = 2x$

$$= \int_0^1 \int_{2x}^{-2x+4} x - y + 20 \quad dy \, dx$$

$$A(x) = \int_{2x}^{-2x+4} x - y + 20 \quad dy$$

$$= \left(xy - \frac{y^2}{2} + 20y \right) \Big|_{2x}^{-2x+4}$$

$$= \left(-2x^2 + 4x - 2(x^2 - 4x + 4) - 40x + 80 \right)$$

$$- \left(\cancel{2x^2} - \cancel{2x^2} + 40x \right)$$

$$= -4x^2 - 84x + 72$$

$$\Rightarrow \text{vol.} = \int_0^1 -4x^2 - 84x + 72 \, dx$$

$$= \left(-\frac{4}{3}x^3 - 42x^2 + 72x \right) \Big|_0^1$$

$$= -\frac{4}{3} - 42 + 72 = \frac{86}{3}.$$