Name: Solution

PID:

Section:

Question	Points	Score
1	10	
2	12	
3	6	
4	12	
Total:	40	

1. Write your Name, PID, and Section on the front of your Blue Book.
2. Write the Version of your exam on the front of your Blue Book.
3. Hand in the first page of the exam with your Blue book.
4. No calculators or other electronic devices are allowed during this exam.
5. You may use one page of notes, but no books or other assistance during this exam.
6. Read each question carefully, and answer each question completely.
7. Write your solutions clearly in your Blue Book
   (a) Carefully indicate the number and letter of each question.
   (b) Present your answers in the same order they appear in the exam.
   (c) Start each question on a new page.
8. Show all of your work; no credit will be given for unsupported answers.

1. Suppose $f(1,3) = 1$, $f_x(1,3) = 1$ and $f_y(1,3) = 2$.
   (a) (6 points) Find equation of the tangent plane of graph of f at $(1,3,1)$.
   (b) (4 points) Use linear approximation to estimate $f(1.1, 2.9)$.
2. Suppose $\nabla g(1,2,3) = (2,1,1)$ and $g(1,2,3) = 1$.
   (a) (4 points) Find equation of the tangent plane of the level surface $g(x,y,z) = 1$ at $(1,2,3)$.
   (b) (4 points) Find the maximum rate of change of g at $(1,2,3)$.
   (c) (4 points) Determine whether g is increasing in the direction of $(-1,2,-1)$. Briefly justify your answer.
3. (6 points) Suppose $z = x\cos y$, $x = u^2 - 2v^2$, and $y = 3u + v$. Find $\partial z/\partial v$. (You can write your answer in terms of x, y, u, and v.)
4. Suppose $h(x,y) = x^3 + y^3 - 3x - 3y^2 + 1$.
   (a) (6 points) Find critical points of h.
   (b) (6 points) For any critical point p_0, determine whether h has a local maximum or local minimum at p_0 or p_0 is a saddle point.

Good Luck!

1. (a) $z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$\Rightarrow \quad z - 1 = (1)(x-1) + (2)(y-3)$

$\Rightarrow \quad z = x + 2y + 1 - 1 - 6 = x + 2y - 6$.

$$\boxed{z - x + 2y = -6}$$

(b) $L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$\Rightarrow L(x,y) = 1 + (1)(x-1) + (2)(y-3)$

$\Rightarrow f(1.1, 2.9) \approx L(1.1, 2.9) = 1 + 0.1 + (2)(-0.1)$

$= 0.9$.

2. (a) $\nabla g(1,2,3)$ is a normal vector of the tangent plane of the level surface $g(x,y,z) = g(1,2,3) = 1$ at the point $(1,2,3)$. So here is an equation of the tangent plane:

$$2(x-1) + 1(y-2) + 1(z-3) = 0$$

(Recall that equation of a plane with normal vector (a,b,c) passing through (x_0, y_0, z_0) is $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$.)

So $2x + y + z = 2 + 2 + 3 = 7$. We get

$$\boxed{2x + y + z = 7}$$

(b) Max. rate of change of g at $(1,2,3) = \|\nabla g(1,2,3)\|$

$$= \sqrt{2^2 + 1^2 + 1^2}$$

$$= \sqrt{6}.$$

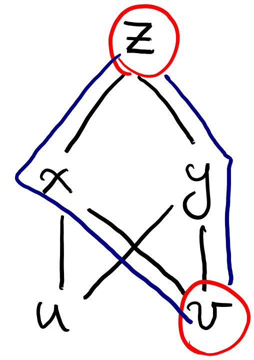
(c) We need to find out sign of the directional derivative of g at $(1,2,3)$ in the direction of $\vec{v} = (-1, 2, -1)$.

$$D_{\vec{u}}\, g(1,2,3) = \frac{\nabla g(1,2,3) \cdot \vec{v}}{\|\vec{v}\|} = \frac{(2,1,1) \cdot (-1,2,-1)}{\sqrt{(-1)^2 + (2)^2 + (-1)^2}}$$

$$= \frac{-2+2-1}{\sqrt{6}} = \frac{-1}{\sqrt{6}} < 0$$

So g is decreasing in the direction of $(-1,2,-1)$ at $(1,2,3)$.

3. $z = x\cos y$, $x = u^2 - 2v^2$, $y = 3u + v$



$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial x} = \cos y \quad , \quad \frac{\partial z}{\partial y} = -x\sin y ,$$

$$\frac{\partial x}{\partial v} = 2u \quad , \quad \frac{\partial y}{\partial v} = 1$$

So $\underline{\dfrac{\partial z}{\partial v} = 2u\cos y - x\sin y}$.

4.(a) This function is everywhere differentiable. So to find its critical points, we should solve $\nabla h(x,y) = (0,0)$.

$\nabla h(x,y) = (h_x, h_y) = (3x^2-3, 3y^2-6y) = (0,0)$

- $3x^2-3=0$ implies $x^2=1$, and so $x=\pm 1$.
- $3y^2-6=0$ implies $y^2=2$, and so $y=\pm\sqrt{2}$

So there are four critical points $(1,\sqrt{2})$, $(-1,\sqrt{2})$, $(1,-\sqrt{2})$ and $(-1,-\sqrt{2})$.

(b) $h_{xx} = 6x$, $h_{xy} = 0$, $h_{yy} = 6y-6$. Since all the 2^{nd} order partial derivatives are continuous, we can use the 2^{nd} derivative test.

Let $D = h_{xx}\cdot h_{yy} - h_{xy}^2 = 36\,x(y-1)$.

Critical points	Sign of D	Sign of f_{xx}	Conclusion
$(1, \sqrt{2})$	$+$	$+$	local min
$(-1, \sqrt{2})$	$-$		saddle point
$(1, -\sqrt{2})$	$-$		saddle point
$(-1, -\sqrt{2})$	$+$	$-$	local max