Name: Solution

PID:

Section:

Question	Points	Score
1	8	
2	10	
3	10	
4	12	
Total:	40	

1. Write your Name, PID, and Section on the front of your Blue Book.
2. Write the Version of your exam on the front of your Blue Book.
3. No calculators or other electronic devices are allowed during this exam.
4. You may use one page of notes, but no books or other assistance during this exam.
5. Read each question carefully, and answer each question completely.
6. Write your solutions clearly in your Blue Book
   (a) Carefully indicate the number and letter of each question.
   (b) Present your answers in the same order they appear in the exam.
   (c) Start each question on a new page.
7. Show all of your work; no credit will be given for unsupported answers.

1. A particle's position function is $\mathbf{r}(t) = \langle \ln t, t^2, 2t+1\rangle$.
   (a) (3 points) Find the particle's velocity $\mathbf{v}(t)$ and its acceleration $\mathbf{a}(t)$.
   (b) (2 points) Find the particle's speed $\|\mathbf{v}(t)\|$ as a function of t. (Simplify your answer)
   (c) (3 points) Find the total distance traveled by the particle during the time interval $1 \le t \le 2$.

2. Evaluate the limit or determine that it does not exist.
   (a) (5 points) $\lim_{(x,y)\to(1,0)} (x^2-1)\cos\left(\frac{1}{(x-1)^2+y^2}\right)$.
   (b) (5 points) $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$.

3. Let $f(x,y) = \cos(x^2+y)$ and $P_0 = (1, \pi/2-1, 0)$.
   (a) (3 points) Find $\nabla f(x,y)$.
   (b) (5 points) Find the equation of the tangent plane of $z = f(x,y)$ at P_0.
   (c) (2 points) Find the maximum rate of increase of f at $(1, \pi/2-1)$.

4. Answer the following questions with short justifications:
   (a) (2 points) Suppose $\nabla f(1,2) = \langle -1, 3\rangle$ for some function f. Is f increasing or decreasing in the direction of $\mathbf{v} = \langle 2, 1\rangle$.
   (b) (2 points) Find a normal vector of the tangent plane of the hyperboloid $\frac{x^2}{4} + y^2 - \frac{z^2}{9} = 1$ at $(2,1,3)$.
   (c) (3 points) Find $\partial z/\partial y$ where $z = f(x,y)$ satisfies $e^{xy} + \sin(xz) + y = 0$. (Your answer can be in terms of x, y, and z.)
   (d) (3 points) Let $x = s+t$ and $y = s-t$. Show that for any differentiable function $f(x,y)$ we have $f_x^2 - f_y^2 = f_s f_t$.
   (e) (2 points) We are told that the velocity of a particle is $\langle -1, -2\rangle$ and its acceleration is $\langle -3, 1\rangle$. Is the particle slowing down or speeding up?

a) $D_{\vec{u}} f(P) = \frac{1}{\|\vec{v}\|} \vec{v} \cdot \nabla f(p)$; $\nabla f(1,2) \cdot \vec{v} = \langle -1, 3\rangle \cdot \langle 2, 1\rangle = -2+3 = 1 > 0$

$\Rightarrow$ increasing.

b) $\nabla F(2,1,3)$ is a normal vector of the plane tangent to the level surface $F(2,1,3) = 1$. $\nabla F = \langle \frac{x}{2}, 2y, -\frac{2z}{9}\rangle \Rightarrow \vec{n} = \langle 1, 2, -\frac{2}{3}\rangle$.

Good Luck!

1. $\vec{v}(t) = \vec{r}\,'(t) = \langle \frac{1}{t}, 2t, 2 \rangle$.

$\vec{a}(t) = \vec{v}\,'(t) = \langle \frac{-1}{t^2}, 2, 0 \rangle$.

$\|\vec{v}(t)\| = \sqrt{\frac{1}{t^2} + 4t^2 + 4} = \sqrt{(\frac{1}{t} + 2t)^2} = |\frac{1}{t} + 2t|$

Total distance $= \int_1^2 \|\vec{v}(t)\|\, dt = \int_1^2 \frac{1}{t} + 2t\, dt$

$= (\ln t + t^2)\Big|_1^2 = \ln 2 + 3$.

2. (a) $0 \le \left| (x^2-1)\cos\left(\frac{1}{(x-1)^2+y^2}\right)\right| \le |x^2-1|$

$\lim\limits_{(x,y)\to(1,0)} 0 = \lim\limits_{(x,y)\to(1,0)} |x^2-1| = 0$

So by Squeeze theorem

$\lim\limits_{(x,y)\to(1,0)} \left|(x^2-1)\cos\left(\frac{1}{(x-1)^2+y^2}\right)\right| = 0$. And so

$\lim\limits_{(x,y)\to(1,0)} (x^2-1)\cos\left(\frac{1}{(x-1)^2+y^2}\right) = 0$.

(b) Limit does NOT exist. Since along the line $y=x$

$\lim\limits_{x\to 0} \frac{(x)(x^2)}{x^2+x^4} = \lim\limits_{x\to 0} \frac{x}{1+x^2} = 0$,

and along $x = y^2$

$\lim\limits_{y\to 0} \frac{(y^2)(y^2)}{(y^2)^2+y^4} = \lim\limits_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2}$. And $0 \ne 1/2$.

3. ⓐ $\nabla f = \langle f_x, f_y \rangle = \langle -2x\sin(x^2+y), -\sin(x^2+y) \rangle$

ⓑ $z = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + z_0$

$f_x(1, \pi/2 - 1) = -2 \quad , \quad f_y(1, \pi/2 - 1) = -1$

$\Rightarrow z = -2(x-1) - (y - \pi/2 + 1)$

$\Rightarrow z + 2x + y = 1 + \pi/2 .$

ⓒ The max. rate of increase $= \|\nabla f(x_0,y_0)\| = \sqrt{(-2)^2 + (-1)^2} = \sqrt{5}.$

4. ⓒ $z = f(x,y)$ satisfies $F(x,y,z) = c \Rightarrow \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$

$F(x,y,z) = e^{xy} + \sin(xz) + y$

$\Rightarrow F_y = x e^{xy} + 1$ and $F_z = x\cos(xz)$

$\Rightarrow \frac{\partial z}{\partial y} = -\frac{x e^{xy} + 1}{x\cos(xz)} .$

ⓓ $f_s = \frac{\partial x}{\partial s} f_x + \frac{\partial y}{\partial s} f_y = f_x + f_y$

$f_t = \frac{\partial x}{\partial t} f_x + \frac{\partial y}{\partial t} f_y = f_x - f_y$

$\Rightarrow f_s f_t = (f_x + f_y)(f_x - f_y) = f_x^2 - f_y^2$

ⓔ $\frac{d}{dt}(\vec{v}(t) \cdot \vec{v}(t)) = 2\, \vec{a}(t) \cdot \vec{v}(t)$

$\langle -1, -2 \rangle \cdot \langle -3, 1 \rangle = 3 - 2 = 1 > 0$

$\Rightarrow$ The derivative of the square of speed is positive

$\Rightarrow$ speed is increasing